2007 International Symposium on Nonlinear Theory and its Applications
NOLTA'07, Vancouver, Canada, September 16-19, 2007

On Analysis and Control of the Cart-Pendulum System Modeled by Discrete Mechanics

Tatsuya KAI† and Yohei YAMAMOTO†

†Dept. of Mechanical Engineering, Grad. School of Engineering, Osaka University, JAPAN
Email: {kai, yamamoto}@watt.mech.eng.osaka-u.ac.jp

Abstract—This paper deals with the cart-pendulum modeled by discrete mechanics, which is known as a good discretizing tool for mechanical systems. We first derive the discrete model of the cart-pendulum. Next, the discrete cart-pendulum is analyzed from the aspect of solvability. We then derive a control algorithm to stabilize the discrete cart-pendulum based on its linear approximation system. Finally, simulations are shown to demonstrate effectiveness of the proposed algorithm.

1. Introduction

Mechanical systems are normally expressed by ordinary or partial differential equations as continuous-time models. Therefore, when we simulate these continuous-time systems by computers, we have to approximate them by discrete-time models with discretizing tools such as Runge-Kutta method. However, it is well known that discretization causes loss of properties of the original continuous-time systems and numerical errors.

On the contrary, *discrete mechanics*, which is a direct discretizing technique for mechanical systems, has been researched recently [1, 2, 3]. It is known that discrete mechanics has some interesting properties: (i) it can describe energies for conservative/dissipative systems with less errors, (ii) some laws of physics such as Noether's theorem are satisfied. It is expected that discrete mechanics is available for designing controllers with a high affinity for computers. However, there are few researches on control of mechanical systems via discrete mechanics [4, 5].

This paper is concerned with the cart-pendulum modeled by discrete mechanics. We analyze the implicit discrete-time system of the inverted pendulum from the aspect of solvability. Then, we derive a control law of the discrete-time system based on its linear approximation system and show some simulation results to demonstrate effectiveness of proposed algorithm.

2. Discrete Mechanics

This section sums up basic concepts of discrete mechanics [1, 2, 3]. Let Q be a configuration manifold and $q \in \mathbf{R}$ be a generalized coordinate of Q. We also refer to T_qQ as the tangent space of Q at a point $q \in Q$ and $\dot{q} \in T_qQ$ denotes a generalized velocity. Moreover, we consider a time-invariant Lagrangian as $L(q, \dot{q}) : TQ \to \mathbf{R}$. We first explain about the discretization method. The time variable $t \in \mathbf{R}$ is discretized as $t = kh$ $(k = 0, 1, 2, \cdots)$ by using a sampling interval $h > 0$. We denote q_k as a point of Q at the time step k, that is, a curve on Q in the continuous setting is represented as a sequence of points $q^d := \{q_k\}_{k=1}^N$ in the discrete setting. The transformation method of discrete mechanics is carried out by the replacement:

$$q \approx (1-\alpha)q_k + \alpha q_{k+1}, \quad \dot{q} \approx \frac{q_{k+1} - q_k}{h}, \tag{1}$$

where q is expressed as a internally dividing point of q_k and q_{k+1} with a ratio α $(0 < \alpha < 1)$. We then define *a discrete Lagrangian*:

$$L_\alpha^d(q_k, q_{k+1}) := hL\left((1-\alpha)q_k + \alpha q_{k+1}, \frac{q_{k+1} - q_k}{h}\right), \tag{2}$$

and *a discrete action sum*:

$$S_\alpha^d(q_0, q_1, \cdots, q_N) = \sum_{k=0}^{N-1} L_\alpha^d(q_k, q_{k+1}). \tag{3}$$

We next summarize the discrete equations of motion. Consider a variation of points on Q as $\delta q_k \in T_{q_k}Q$ $(k = 0, 1, \cdots, N)$ with the fixed condition $\delta q_0 = \delta q_N = 0$. In analogy with the continuous setting, we define a variation of the discrete action sum (3) as

$$\delta S_\alpha^d(q_0, q_1, \cdots, q_N) = \sum_{k=0}^{N-1} \delta L_\alpha^d(q_k, q_{k+1}). \tag{4}$$

The discrete Hamilton's principle states that *only a motion that makes the discrete action sum (3) stationary is realized*. Calculating (4), we have

$$\delta S_\alpha^d = \sum_{k=1}^{N-1} \{D_1 L_\alpha^d(q_k, q_{k+1})\delta q_k + D_2 L_\alpha^d(q_{k-1}, q_k)\}\delta q_k, \tag{5}$$

where D_1 and D_2 denotes the partial differential operators with respect to the first and second arguments, respectively. Consequently, from the discrete Hamilton's principle and (5), we obtain *the discrete Euler-Lagrange equations*:

$$\begin{aligned} D_1 L_\alpha^d(q_k, q_{k+1}) + D_2 L_\alpha^d(q_{k-1}, q_k) = 0,& \\ k = 1, \cdots, N-1.& \end{aligned} \tag{6}$$

It turns out that (6) is represented as difference equations which contains three points q_{k-1}, q_k, q_{k+1}, and we need q_0, q_1 as an initial condition when we simulate (6).

3. Discrete Cart-Pendulum System

This section derives the discrete model of the cart-pendulum system as shown in Fig. 1. Let $\theta \in \mathbf{S} := (-\pi, \pi]$ be the angle of the pendulum and $z \in \mathbf{R}$ be the position of the cart. We set parameters of the systems: the mass of the pendulum m, the mass of the cart M, and the length of the pendulum l. For sake of simplicity, we do not take account of frictions. The Lagrangian of this system is given by

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{\theta}\dot{z}\cos\theta + \frac{1}{2}(m+M)\dot{z}^2 - mgl\cos\theta. \quad (7)$$

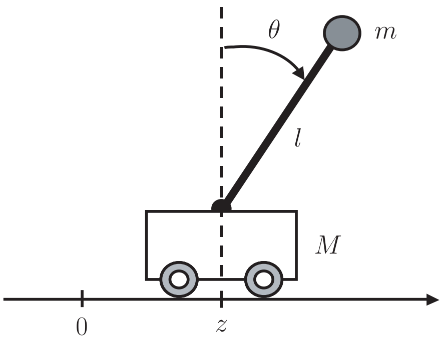

Fig. 1 : The Cart-Pendulum System

From the discrete Lagrangian with $q_k = [\ \theta_k \ \ z_k\ ]^\mathsf{T}$:

$$\begin{aligned} L^d(q_k, q_{k+1}) &= \frac{m+M}{2h}(z_{k+1}-z_k)^2 + \frac{ml^2}{2h}(\theta_{k+1}-\theta_k)^2 \\ &+ \frac{ml}{h}\cos\{(1-\alpha)\theta_k + \alpha\theta_{k+1}\}(z_{k+1}-z_k)(\theta_{k+1}-\theta_k) \\ &- mglh\cos\{(1-\alpha)\theta_k + \alpha\theta_{k+1}\}, \end{aligned} \quad (8)$$

and the control input to the cart u_k, we can derive the discrete Euler-Lagrange equation of the cart-pendulum system as

$$\begin{aligned} &- ml(1-\alpha)(\theta_{k+1}-\theta_k)(z_{k+1}-z_k)\sin\{(1-\alpha)\theta_k + \alpha\theta_{k+1}\} \\ &- ml\cos\{(1-\alpha)\theta_k + \alpha\theta_{k+1}\}(z_{k+1}-z_k) \\ &- ml^2(\theta_{k+1}-\theta_k) + mgl(1-\alpha)h^2\sin\{(1-\alpha)\theta_k + \alpha\theta_{k+1}\} \\ &- ml\alpha(\theta_k-\theta_{k-1})(z_k-z_{k-1})\sin\{(1-\alpha)\theta_{k-1} + \alpha\theta_k\} \\ &+ ml\cos\{(1-\alpha)\theta_{k-1} + \alpha\theta_k\}(z_k-z_{k-1}) \\ &+ ml^2(\theta_k-\theta_{k-1}) + mgl\alpha h^2\sin\{(1-\alpha)\theta_{k-1} + \alpha\theta_k\} = 0, \end{aligned} \quad (9)$$

and

$$\begin{aligned} &- (m+M)(z_{k+1}-z_k) \\ &- ml(\theta_{k+1}-\theta_k)\cos\{(1-\alpha)\theta_k + \alpha\theta_{k+1}\} \\ &+ (m+M)(z_k-z_{k-1}) \\ &+ ml(\theta_k-\theta_{k-1})\cos\{(1-\alpha)\theta_k + \alpha\theta_{k-1}\} + hu_k = 0. \end{aligned} \quad (10)$$

Substituting $\theta_{k-1} = \theta_k = \theta_{k+1}$, $z_{k-1} = z_k = z_{k+1}$ and $u_k = 0$ into (9) and (10), we have $\sin\theta_k = 0$. Therefore, the equilibria of the discrete cart-pendulum system are $(\theta_k, z_k) = (0, z^e)$, (π, z^e), $\forall z^e \in \mathbf{R}$, that is, they correspond with those of the usual cart-pendulum in the continuous setting. Finally, we calculate the linear approximation system that behaves around the equilibrium $\theta_k = 0$. Considering θ_{k-1}, θ_k, $\theta_{k+1} \approx 0$ for (9) and (10), we obtain the linear approximation system as

$$\begin{aligned} &- ml(z_{k+1}-z_k) + mgl(1-\alpha)h^2\{(1-\alpha)\theta_k + \alpha\theta_{k+1}\} \\ &+ ml(z_k-z_{k-1}) - ml^2(\theta_{k+1}-\theta_k) + ml^2(\theta_k-\theta_{k-1}) \\ &+ mgl\alpha h^2\{(1-\alpha)\theta_{k-1} + \alpha\theta_k\} = 0 \end{aligned} \quad (11)$$

and

$$\begin{aligned} &- (m+M)(z_{k+1}-z_k) - ml(\theta_{k+1}-\theta_k) \\ &+ (m+M)(z_k-z_{k-1}) + ml(\theta_k-\theta_{k-1}) + hu_k = 0. \end{aligned} \quad (12)$$

4. Solvability Analysis

By using appropriate functions f and g, we can rewrite (9) and (10) as

$$f(\theta_{k-1}, \theta_k, \theta_{k+1}, z_{k-1}, z_k, u_k) = 0, \quad (13)$$

$$z_{k+1} = g(\theta_{k-1}, \theta_k, \theta_{k+1}, z_{k-1}, z_k, u_k). \quad (14)$$

We can see that (14) is an explicit function for z_{k+1}, however, (13) is an implicit function for θ_{k+1}. In general, systems modeled by discrete mechanics contain some implicit equations, hence we have to treat implicit nonlinear discrete-time systems. In this section, we investigate solvability of the discrete cart-pendulum system. We first explain some concepts of solvability for implicit nonlinear discrete-time systems:

$$f_k(x_k, x_{k+1}, u_k) = 0, \ \ k = 0, 1, \cdots, N-1, \quad (15)$$

where $k \in \{0, \cdots, N\}$ is a time step, $x_k \in \mathbf{R}^n$ is a state, $u_k \in \mathbf{R}^m$ is an input and $f_k : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^r$ is a nonlinear function. We often find implicit nonlinear discrete-time systems in economics [6]. The class of implicit nonlinear discrete-time systems contains descriptor systems and is the largest in all of the discrete-time control systems. Luenberger [6] and Fliegner et al. [7, 8] discussed *solvability* for such systems. If a given pair of a state sequence $x := (x_0, x_1, \cdots, x_N)$ and an input sequence $u := (u_0, u_1, \cdots, u_{N-1})$ satisfies all the equations of (15), it is called *admissible*. *A solvability matrix* for a admissible pair (x, u) is defined by

$$\begin{aligned} F_{(x,u)}(0, N) &:= \begin{bmatrix} G_0 & H_1 & & & \\ & G_1 & H_2 & & \\ & & & \ddots & \\ & & & G_{N-1} & H_N \end{bmatrix}, \\ G_i &:= \left.\frac{\partial f_i}{\partial x_i}\right|_{(x_i, x_{i+1}, u_i)}, \ \ i = 0, \cdots, N-1, \\ H_i &:= \left.\frac{\partial f_i}{\partial x_{i+1}}\right|_{(x_{i-1}, x_i, u_{i-1})}, \ \ i = 1, \cdots, N. \end{aligned} \quad (16)$$

Solvability of the system (15) is defined as follows [8].

Definition 1: The implicit nonlinear discrete-time system (15) is said to be *solvable* if the solvability matrix (16) has a row full-rank for any admissible pair (x, u). ∎

In the simplest terms, solvability means the existence of x_{k+1} for given x_k and u_k. In order to check solvability of a given system, *the shuffle algorithm* based on the implicit theorem is introduced [8]. We can investigate solvability by the rank of a finally obtained system in the algorithm. Because of space limitations, we omit its details (see [8]).

We now check solvability of the discrete cart-pendulum system. We can get the following by the shuffle algorithm.

Proposition 1: Assume that the sampling time is sufficient small, i.e., $h \ll 1$. Then, the discrete cart-pendulum system (9), (10) is solvable at any point (θ, z). ∎

We next consider solvability when the system behaves around the equilibrium $\theta_k = 0$ and the sampling time h is not subject to restrictions. The following can be derived.

Proposition 2: The discrete cart-pendulum system (9), (10) is solvable around neighborhood of the equilibrium point $(0, z)$ if

$$-Ml + (m + M)g(1 - \alpha)\alpha h^2 \neq 0 \tag{17}$$

holds for its parameters. ∎

From Proposition 2, if the sampling time h satisfies

$$h \neq \sqrt{\frac{Ml}{(m + M)g(1 - \alpha)\alpha}}, \tag{18}$$

the discrete cart-pendulum (9) and (10) is always solvable around the equilibrium $\theta_k = 0$. Since (17) is a sufficient condition, the system has a possibility of solvability though (17) fails.

5. Stabilization of Discrete Cart-Pendulum System

This section gives a stabilizing controller for the discrete cart-pendulum system and shows some simulations. We first set a state variable as $x_k = [\, x_k^1 \;\; x_k^2 \;\; x_k^3 \;\; x_k^4 \,]^{\mathrm{T}} = [\, \theta_k \;\; \theta_{k+1} \;\; z_k \;\; z_{k+1} \,]^{\mathrm{T}}$. From (11) and (12), we then obtain the discrete-time linear control system:

$$x_{k+1} = Ax_k + Bu_k, \tag{19}$$

where

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & \dfrac{2Ml + (m+M)g\{(1-\alpha)^2 + \alpha^2\}h^2}{Ml - (m+M)g(1-\alpha)\alpha h^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \dfrac{-mlgh^2}{Ml - (m+M)g(1-\alpha)\alpha h^2} & -1 & 2 \end{bmatrix},$$

$$B := \begin{bmatrix} 0 \\ \dfrac{-h}{Ml - (m+M)g(1-\alpha)\alpha h^2} \\ 0 \\ \dfrac{lh - (m+M)g(1-\alpha)\alpha h^3}{Ml - (m+M)g(1-\alpha)\alpha h^2} \end{bmatrix}.$$

By solving discrete-time optimal regulator problem for (19), we design a controller in the form

$$u_k = K_1\theta_{k-1} + K_2\theta_k + K_3 z_{k-1} + K_4 z_k, \tag{20}$$

where K_i $(i = 1, \cdots, 4)$ are gain matrices. An algorithm to stabilize the discrete cart-pendulum system is as follows.

Algorithm 1:

Step 0 Decide initial states $\theta_0, \theta_1, z_0, z_1$.

Step 1 Set $k = 1$. Derive θ_2 by solving

$$f(\theta_0, \theta_1, \theta_2, z_0, z_1, u_1) = 0. \tag{21}$$

with Newton method. Then, calculate z_2 by

$$z_2 = g(\theta_0, \theta_1, \theta_2, z_0, z_1, u_1). \tag{22}$$

Reset $k = 2$ and go to Step 2.

Step l Set $k = l$ $(l = 2, 3, \cdots)$. Derive θ_{l+1} by solving

$$f(\theta_{l-1}, \theta_l, \theta_{l+1}, z_{l-1}, z_l, u_l) = 0 \tag{23}$$

with Newton method. Then, calculate z_{l+1} by

$$z_{l+1} = g(\theta_{l-1}, \theta_l, \theta_{l+1}, z_{l-1}, z_l, u_l) \tag{24}$$

If $l < N$, then reset $k = l + 1$ and go to Step $l + 1$. If $l = N$, the algorithm is over. ∎

We now show simulations of the discrete cart-pendulum by using Algorithm 1. The parameters are set as $m = 0.04$ [kg], $M = 1.00$ [kg], $l = 0.15$ [m], $\alpha = 0.5$. The weight matrices for the discrete-time optimal regulator problem are set as $Q = \mathrm{diag}(1, 0, 1.0, 5.0, 5.0)$, $R = 0.005$. The initial states are given as $\theta_0 = \pi/6$ [rad], $\theta_1 = \pi/6$ [rad], $z_0 = 0.1$ [m], $z_1 = 0.1$ [m].

Fig. 3–5 depict time series of θ and z with the sampling times $h = 0.05,\ 0.1,\ 0.5,\ 1.0$, respectively. Note that there is no meaning of the dashed lines connecting the dots in Fig. 3–5. From these figures, it is confirmed that the discrete cart-pendulum is stabilized at any sampling time. It is interesting that stabilization is performed at a large sampling time such as $h = 1.0$. Therefore, we can say that it is appropriate to design controllers for systems based on their linear approximations in discrete mechanics.

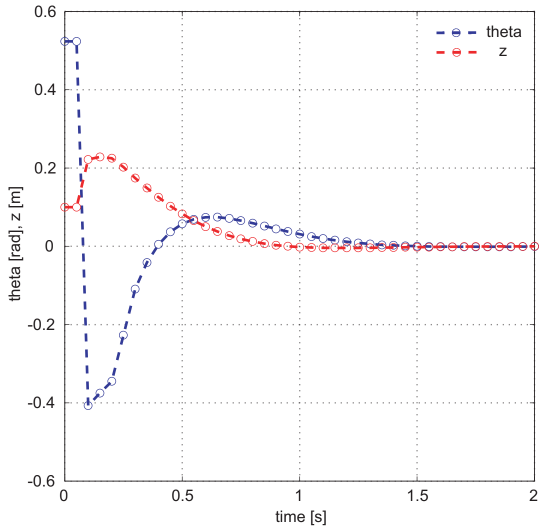

Fig. 2 : Time Responses of θ_k and z_k $(h = 0.05)$

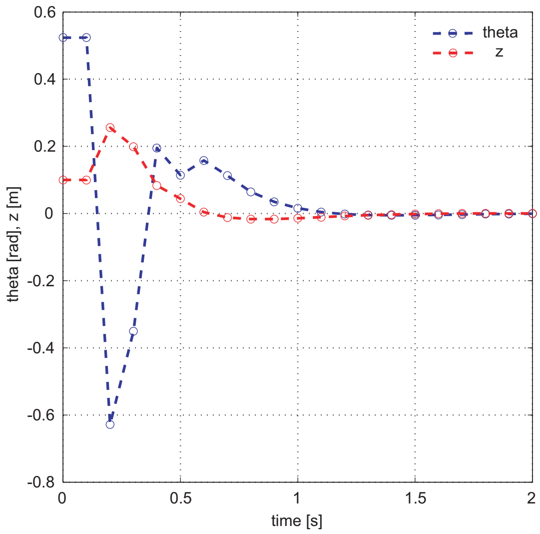

Fig. 3 : Time Responses of θ_k and z_k $(h = 0.1)$

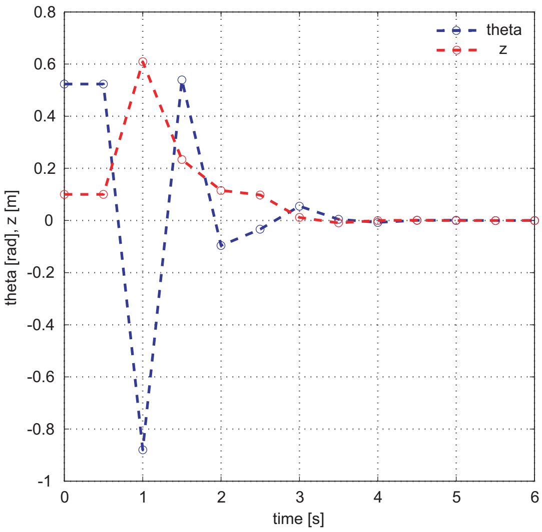

Fig. 4 : Time Responses of θ_k and z_k $(h = 0.5)$

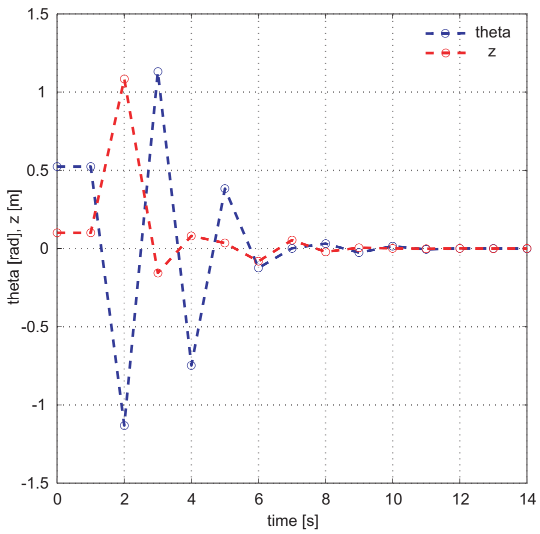

Fig. 5 : Time Responses of θ_k and z_k $(h = 1.0)$

6. Conclusion

This paper has analyzed the discrete cart-pendulum system from the viewpoint of solvability of implicit nonlinear discrete-time systems. Moreover, we have proposed a stabilizing algorithm for the system and confirmed its effectiveness by some simulations. We can say that an application of discrete mechanics to control theory has been performed through the cart-pendulum system in this paper. However, in order to apply discrete mechanics to control of various mechanical systems, there is still room to study due to difficulties of implicit nonlinear discrete-time systems.

Our future work is as follows: (i) solvability analysis of general implicit nonlinear discrete-time systems modeled by discrete mechanics, (ii) controller design by model predictive control and iterative learning control, (iii) experiment of the cart-pendulum system.

References

[1] J. E. Marsden, G. W. Patrick and S. Shkoller, “Multisymplectic Geometry, Variational Integrators and Nonlinear PDEs,” *Comm. in Math. Phys.*, vol.199, pp.351–395, 1998.

[2] C. Kane, J. E. Marsden, M. Ortiz and M. West, “Variational Integrators and the Newmark Algorithm for Conservative and Dissipative Mechanical Systems,” *Int. J. for Numer. Meth. in Engineering*, vol.49, pp.1295–1325, 2000.

[3] J. E. Marsden and M. West, “Discrete Mechanics and Variational Integrators,” *Acta Numerica*, vol.10, pp.3571-5145, 2001.

[4] A. M. Bloch, M. Leok, J. E. Marsden and D. V. Zenkov, “Controlled Lagrangians and Stabilization of the Discrete Cart-Pendulum System,” *in Proc. of 44th IEEE CDC-ECC*, Seville, Spain, pp.6579-6584, 2005.

[5] A. M. Bloch, M. Leok, J. E. Marsden and D. V. Zenkov, “Controlled Lagrangians and Potential Shaping for Stabilization of the Discrete Mechanical Systems,” *in Proc. of 45th IEEE CDC*, San Diego, USA, pp.3333-3338, 2006.

[6] D. G. Luenberger, “Non-linear Descriptor Systems,” *J. Econom. Dynam. Cont.*, vol.1, pp.219-242, 1979.

[7] T. Fliegner, H. Nijmeijer and Ü. Kotta, “Some Aspects of Nonlinear Discrete-time Descriptor Systems in Economics,” Predictability and Nonlinear Modelling in Natural Sciences and Economics, Kluwer Academic Publishers, pp.581–590, 1994.

[8] T. Fliegner, Ü. Kotta and H. Nijmeijer, “Solvability and Right-inversion of Implicit Nonlinear Discrete-time Systems,” *SIAM J. Cont. and Optim.*, vol.34, no.6, pp.2092–2115, 1996.