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Oscillation in cyclic coupled systems

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Abstract—We investigate the mechanism of oscillatory solutions in coupled even number of neurons as a ring. In a time domain, these oscillatory solutions show switching patterns between positive and negative values. We calculate the distance between the oscillatory solutions and UPS (Unstable Periodic Solution) generated the Hopf bifurcation on the sections of $x_i = 0$. This result shows that these patterns are formed by the trajectories closing to UPS and staying around it for a long time. We also confirm this phenomenon in a simple electrical circuit using inverting operational amplifiers.

1. Introduction

Systems of coupled oscillators have been widely studied as models of information processing in the brain [1], animal locomotion [2], generating nonlinear phenomena such as chaotic itinerancy [3] and on-off intermittency [4, 5], and so on. In particular, we are interested in neuron models, because many investigators confirm that oscillatory dynamics of neural activity and its synchronization play an important role in information processing in the brain.

We consider unidirectionally coupled ring networks of neurons with inhibitory connections. It is said that the essential dynamics of some biological central pattern generators can be captured by a model consisting of N neurons connected in a ring structure [6–8]. In such a system, it is known that when a number of inhibitory neurons is odd and even, the system has stable and unstable oscillatory modes, respectively [9, 10]. However, we found out the existence of oscillatory modes as a quasi-attractor in a system of even inhibitory neurons [11]. In the previous study [12], we investigated the relationship between these oscillatory modes and the unstable periodic solutions(UPS) generated by the Hopf bifurcations in a large number of neurons. In this study, we show that the trajectories of these oscillatory modes approach the UPS and stay around it for a long time even in a small number of neurons. We also confirm this phenomenon in a simple electrical circuit using inverting operational amplifiers.

2. System Equation

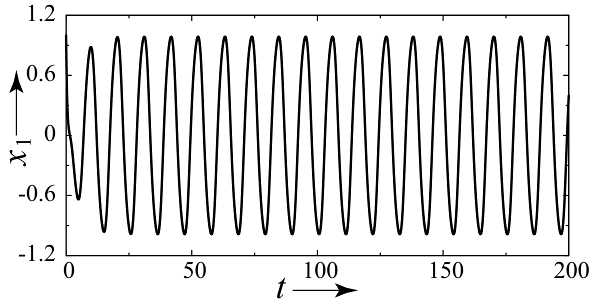
The system equation is described as

$$\tau \frac{dx_i}{dt} = -x_i - cf(x_{i+1}) \qquad (1)$$

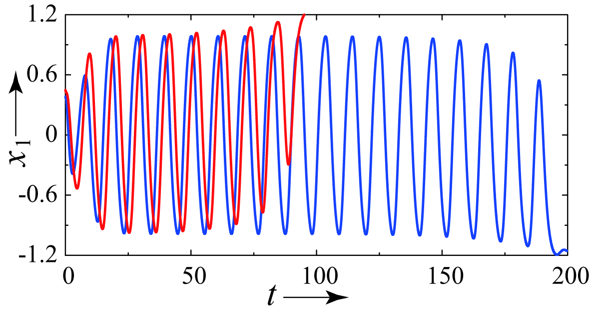
$$(i = 1, 2, \cdots, n, \ x_{n+1} \equiv x_1)$$

where x_i is the state of neuron i, n is the number of neurons(only even number is considered), τ is the time constant fixed as 1.0, c is the coupling coefficient and $f(x)$ is the output function given by $\tan^{-1}(x)$. This type of neuron model is commonly used for controlling locomotion [13–15] and describing oscillatory phenomena [10, 16, 17].

We consider $c > 0$ in Eq. (1), thus Eq. (1) consists of only inhibitory neurons.



(a) UPS.



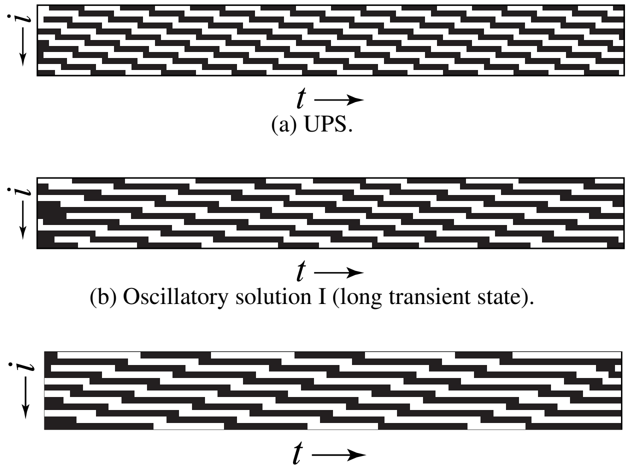
(b) Short(red) and long(blue) transient oscillatory modes from different initial states.

Figure 1: (a)Waveforms of UPS and (b)transient oscillatory modes observed in Eq. (1) for $n = 12$ and $c = 1.4$.

3. Results

3.1. Properties (even n)

The system has a stable equilibrium point at the origin for $c < 1$. At $c = 1$ the pitchfork bifurcation of the equilibrium point occurs and two stable equilibrium points, namely $(x_1, x_2, \cdots, x_n) = (A, -A, \cdots, -A)$ and



(a) UPS.

(b) Oscillatory solution I (long transient state).

(c) Oscillatory solution II (short transient state).

Figure 2: Temporal switching patterns for $c = 1.4$ and $n = 12$ in Eq. (1). When the sign of x_i is changed, we put black and white for negative and positive values of x_i, respectively.

$(-A, A, \cdots, A)$, are generated. The unstable equilibrium point (the origin) meets several Hopf bifurcations for $c > 1$. Figure 1(a) shows the waveform of an unstable periodic solution(UPS) generated by the first Hopf bifurcation. This UPS is one-dimensionally unstable, however, stably observed in the space of $x_i = -x_{i+n/2}$ [12].

Note that for $c > 1$ only two equilibrium points which are symmetry with respect to the origin are attractors. However, we observe oscillatory solutions for sufficiently large n [11]. In this study, we investigate the mechanism of generating such oscillatory solutions for small number of neurons ($n = 12$). In this case, although the solutions go to one of quiescence attractors in finite time(Fig. 1(b)), we consider that the mechanism of generating oscillatory solutions is the same as the case of large n.

3.2. Computer Simulation (n=12)

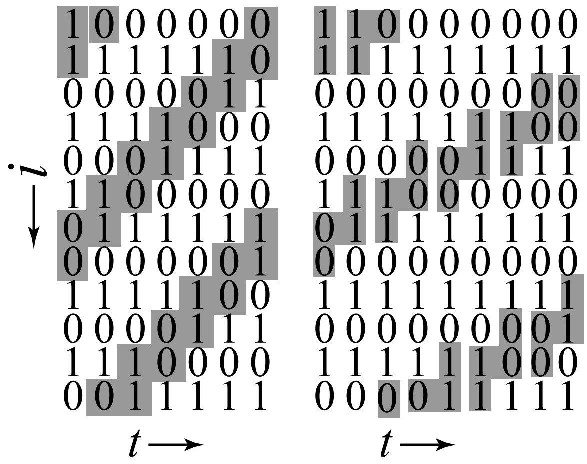
The first and second Hopf bifurcation of the origin occur at $c \simeq 1.155$ and $c \simeq 2.0$, respectively. Thus, we fix the value of c as 1.4 to study the influence of only the first Hopf bifurcation on the oscillatory modes.

Figure 2(a) shows a temporal pattern for the UPS generated by the Hopf bifurcation. We put initial states as $x_i = -x_{i+6}$ ($i = 1, \cdots, 6$), therefore the UPS is stable in this invariant subspace. The temporal pattern is symmetrical black and white. Figure 3(a) shows a symbolized and enlarged temporal pattern in some time interval. We use the following rule:

$$S(x_i) = \begin{cases} 0 & \text{if} \quad x_i < 0 \\ 1 & \text{if} \quad x_i \geq 1 \end{cases} \tag{2}$$

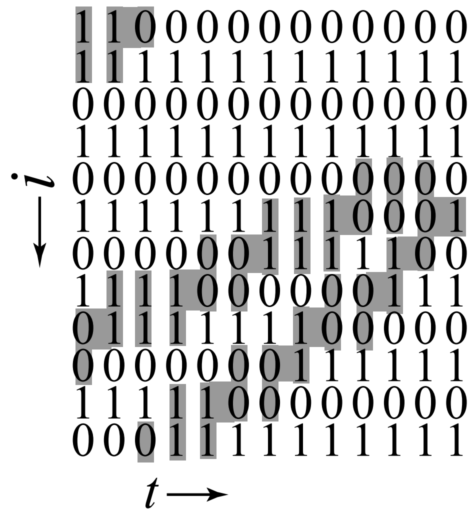
From this figure we can see that the switching the signs of state variables occur simultaneously at two neurons and two propagations proceed at the same speed. On the other hand, the temporal black and white patterns of oscillatory solutions (Figs. 2(b) and (c)) are asymmetrical and the change of the sign occurs for only one neuron at any time (Fig. 3(b)). Finally(Fig. 3(c)), switching rules are broken and the shaded pair approaches each other, and 0 and 1 are alternatively lined. Therefore the trajectory goes to one of quiescence attractors immediately.

We calculate the sum d of the distance on the sections of $x_i = 0$ ($i = 1, 2, \cdots, 12$). More precisely, when the trajectories cross the sections from negative to positive value, we calculate the Euclidean distance between the oscillatory mode and the UPS and add them for all $i's$. Figure 4(a) shows the results from several random initial states. In every case, the distance is decreased once; this means



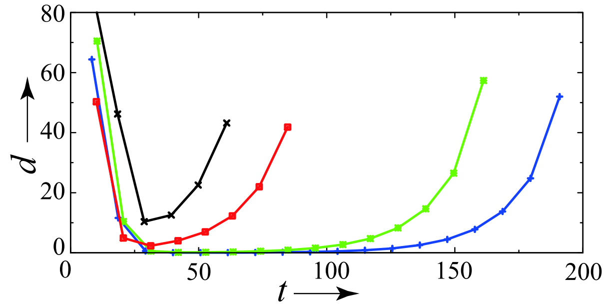
(a) UPS.

(b) Oscillatory solution.

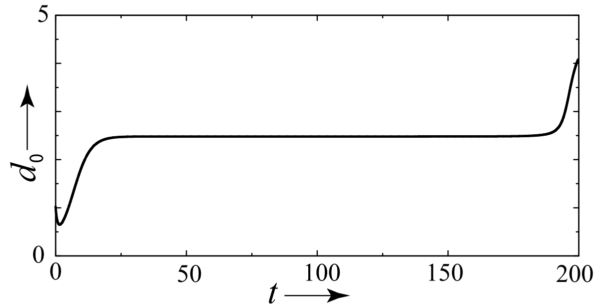


(c) End of oscillatory solution.

Figure 3: Temporal pattern using symbols 1 and 0 for positive and negative values of x_i, respectively. We shade the symbols either $S(x_i(t)) = S(x_{i+1}(t))$ or $S(x_i(t)) \neq S(x_i(t-1))$.



(a) Sum of distance between oscillatory solutions and UPS on the every section of $x_i = 0$ $(i = 1, \cdots, 12)$. Interval of symbols is almost the same as the period of UPS.



(b) Distance of oscillatory solution from the origin.

Figure 4: (a)Distance between oscillatory solutions and UPS, and(b) oscillatory solution and the origin for green curve in (a).

the solution is close to the UPS. For long transient oscillations(green and blue curves) the trajectories stay around the UPS for a long time. Because the UPS is only one-dimensionally unstable, by choosing the appropriate initial states the trajectories approach the UPS. We also calculate the distance d_0 from the origin shown in Fig. 4(b). This solution turns around the origin that corresponds to the long transient.

3.3. Discrete System

We also consider a discrete system defined by

$$y_i(n+1) = ay_i(n)(1 - y_i^2(n)) + cy_{i+1}(n) \quad (i = 1, 2, \cdots, 12,\ y_{13} \equiv y_1) \tag{3}$$

For positive values of a and c, the pitchfork bifurcation occurs at $a + c = 1$. Thus we set $a = 1.1$ and $c = 0.2$. In this case, stable equilibrium points $(y_1,\ y_2,\ \cdots,\ y_{12}) = (B, B, \cdots, B)$ and $(-B, -B, \cdots, -B)$ are generated. We show a temporal switching pattern in Fig. 5. This pattern is also a transient state to one of the two quiescence attractors, however, it is very difficult to distinguish whether transient states or steady states for a large number of cells. We estimate that the key of generating this oscillation is the existence of symmetrical two quiescence attractors with respect to the origin.

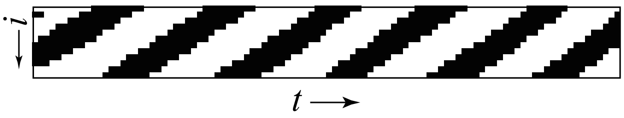


Figure 5: Temporal switching pattern for Eq. (3). The meaning of black and white is the same as Fig. 2.

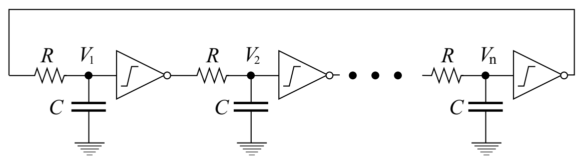


Figure 6: Circuit diagram.

3.4. Circuit Experiment

We consider an analog circuit using inverting amplifiers with diffusive coupling, shown in Fig. 6. The output voltage V_{out} of the inverting amplifiers is a piecewise linear function of the input voltages V_{in} as shown in Eq. (4), in which the operating range (the power supply $\pm V_p$) of the operational amplifiers composing them is taken into account.

$$V_{out} = \begin{cases} -V_p & \text{if} \quad V_{in} > V_p/g, \\ -gV_i & \text{if} \quad -V_p/g \le V_{in} \le V_p/g, \\ V_p & \text{if} \quad V_{in} < -V_p/g \end{cases} \tag{4}$$

where g is the gain of the amplifiers. This simple nonlinearity plays an essential role in the stability and transient phenomena of the system. The circuit equation is given by Eq. (5).

$$CR\frac{dV_i}{dt} = -V_i + V_{out}(V_{i-1}) \quad (i = 1, \cdots, n,\ V_0 \equiv V_n) \tag{5}$$

We observe a long transient oscillation shown in Fig. 7, which last more than ten minutes, in the experiment with the circuit of 40 operational amplifiers diffusively coupled with a time constant (RC) 0.1[sec]. A number of nodes and a piecewise linear system are different from Sec. 3.2, however the essence is not lost, because the keys of generating this oscillation are even number of nodes and the symmetrical property of inverting state variables

The duration t_s of the oscillations approximately obeys a power law distribution, i.e. $f(t_s) \simeq 1/t_s$, not an exponential distribution. We then study the dependence of the transient oscillations on fluctuations in the initial values of the voltages of the nodes and variations in the values of the elements with computer simulation. Negative spatial correlations in the initial voltages make the oscillations less occur and shorten the duration, while positive correlations have little effect. Variations in the elements: the resistance R, the capacitance C and the gain g of the amplifiers also

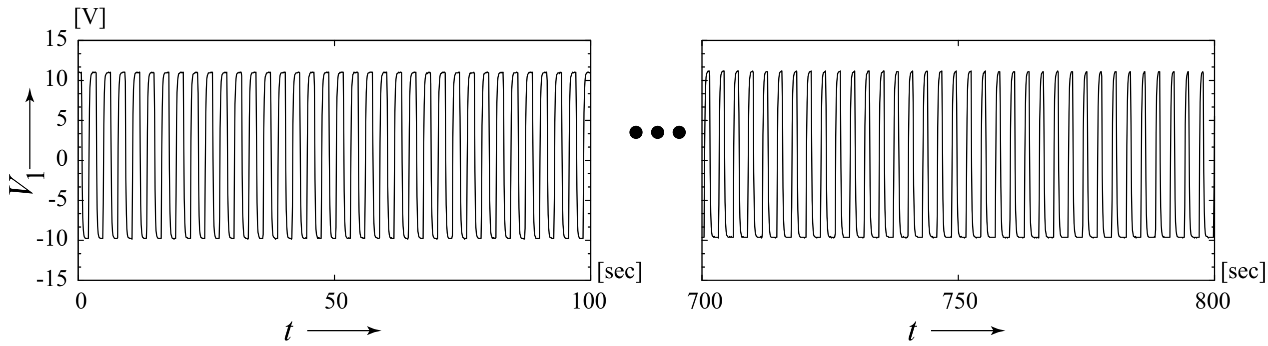


Figure 7: Time series data from experiment.

decrease the occurrence of the oscillations since they break the symmetry of the system. Long oscillations, however, still appear even though the values of the elements vary in the ranges of the order of several hundred times.

4. Conclusions

We investigate the mechanism of oscillatory solutions in coupled even number of neurons as a ring. In a time domain, these oscillatory solutions show switching patterns between positive and negative values. We calculate the distance between the oscillatory solutions and the UPS generated by the Hopf bifurcation on the sections $x_i = 0$. This result shows that these patterns are formed by the trajectories closing to the UPS and staying around it for a long time. We also study the discrete system and the piecewise linear system as the model of an analog circuit. From these results, we consider the key of generating this phenomenon is that the system has the symmetrical property of inverting state variables and two symmetrical equilibrium points with respect to the origin. However, we can say that some perturbations are allowed, because we confirm this phenomenon in a simple electrical circuit using inverting operational amplifiers. It is an open problem to study other coupling methods [18, 19].

References

[1] N. Katayama, M. Nakao, H. Saitoh and M. Yamamoto, Dynamics of a hybrid system of a brain neural network and an artificial nonlinear oscillators, BioSystems, vol.58, pp.249–257, 2000.

[2] J.J. Collins and I. Stewart, Coupled Nonlinear Oscillators and the Symmetries of Animal Gaits, J. Nonlinear Science, vol.3, pp.349–392, 1993.

[3] Focus issue: Chaotic Itineracy, CHAOS, vol.13, no.3, pp. 926–1164, 2003.

[4] H. Fujisaka and T. Yamada, A new intermittency in coupled dynamical systems, Prog. Theor. Phys. vol.74, no.4, pp.918–921, 1985.

[5] H. Fujisaka and T. Yamada, Stability theory of synchronized motion in coupled-oscillator systems.IV, Prog. Theor. Phys. vol.75, no.5, pp.1087–1104, 1986.

[6] G.B. Ermentrout, The behavior of rings of coupled oscillators, J. Math. Biology, vol.23, pp.55–74, 1985.

[7] C.C. Canavier, D.A. Baxter, J.W. Clark and J.H. Byrne, Control of multistability in ring circuits of oscillators, Biol. Cybern. vol.80, pp.87–102, 1999.

[8] C.Y. Park, Y. Hayakawa, K. Nakajima and Y. Sawada, Limit cycles of one-dimensional neural networks with the cyclic connection matrix, IEICE Trans. Fundamentals, vol.E79-A, no.6, pp.752–757, June 1996.

[9] S. Amari, Mathematics of neural networks, Sangyo-Tosho, 1978 (in Japanese).

[10] Y. Nakamura and H. Kawakami, Bifurcation and chaotic attractor in a neural oscillator with three analog neurons, Electronics and Communications in Japan, Part3, vol.83, no.9, pp.104–110, 2000.

[11] T. Ishii and H. Kitajima Oscillation in cyclic coupled neurons, Proc. NCSP'06, Hawaii, U.S.A., pp.97–100, March 2005.

[12] H. Kitajima, T. Ishii and T. Hattori Oscillation mechanism in cyclic coupled neurons, Proc. NOLTA'2006, pp.623–626, Bologna, Italy, Sept. 2006.

[13] A. Fujii, A. Ishiguro, Y. Uchikawa, T. Aoki and P. Eggenberger, Gait control for a legged robot using neural networks with dynamically-rearranging function, IEEJ Trans., vol.119-C, no.12, pp.1567–1572, 1999 (in Japanese).

[14] H. Nagashino, M. Kataoka and Y. Kinouchi, A coupled neural oscillator model for recruitment and annihilation of the degrees of freedom of oscillatory movements, Neurocomputing, vol.26–27, pp.455–462, 1999.

[15] H. Nagashino, Y. Nomura and Y. Kinouchi, A neural network model for quadruped gait generation and transitions, Neurocomputing, vol.38–40, pp.1469–1475, 2001.

[16] H.H. Yang, Some results on the oscillation of neural networks, Proc. NOLTA'95, Las Vegas, U.S.A., pp.239–242, Dec. 1995.

[17] K. Jinno and T. Saito, Analysis and Synthesis of Continuous-Time Hysteretic Neural Networks, IEICE Trans. Fundamentals, vol.J75-A, no.3, pp.552–556, March 1992.

[18] G.N. Borisyuk, R.M. Borisyuk, A.I. Khibnik and D. Roose, Dynamics and bifurcation of two coupled neural oscillators with different connection types, Bull. Math. Biol. vol.57, no.6, pp.809–840, 1995.

[19] D.J. Watts, Six Degrees: The Science of a Connected Age, W.W. Norton & Company, 2003.