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Genetic Algorithm-based Parameter Optimization of Tsallis Distribution and Its Application to Financial Markets

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Abstract—This study proposes a heuristic method on how to realize Genetic Algorithm-based parameter optimization of Tsallis dsitribution, and its application to financial markets. Conventionally, returns distribution is tackled as normal, lognormal, or non-Gaussian stable distribution, as it has been shown in previous researches, these methods are not accurate enough to catch the characteristicses of returns distribution, in some cases, serious biases could be introduced, in estimating Value at Risk, for example. On the other hand, to use mixture distribution has been suggested in our previous work, quantitative analyses have shown that it can catch the charateristicses of returns distribution, such as kurtosis, finite moments, and heavy-tailed behavior well, and it also provides an accurate approximation of original distribution. However, it just catches the characteristicses of the distribution at one special tme span, daily, weekly, monthly, and so on. It is well observed that a returns distribution usually evolves over different time spans, especially on kurtosis, and standard deviation. To grasp the whole picture of returns distribution dynamically, we propose to model it as a Tsallis distribution, whose parameters are optimized by Genetic Algorithm. Since Tsallis distribution can provide a dynamic probability density function which evolves over different time spans, as a dynamic trace for returns distribution's evolution. In our numerical studies, we find that our proposed method works well on tracing the whole evolving picture of returns distribution.

1. Introduction

Recently many researches have focused on how to approximate or identify returns distributions of financial markets. Normal, lognormal, and non-Gaussian stable distribution have been suggested to tackle this problem [1][2][3][4]. As it is pointed out in previous researches [1][5], theses models have their own merits and demerits when applied to real stock markets. But, many quantitative analyses have shown that these models are not supported by the real business realities. Serious biases could be introduced by these models in risk measurement or management, in estimating Value at Risk (VaR), for example. Therefore, a new method suggested in our previous work is to use a mixture distribution to fit a returns distribution. Here mixture distribution is constructed as a weighed sum of several radical distributions, such as normal, or Student-t distribution and so on. It has been shown that the characteristicses of a returns distribution can be caught in a mixture model accurately [5]. Thus, mixture distribution method provides more accurate estimates for risk management, such as estimates of the VaR. However, quantitative researches have shown that a returns distribution usually evolves over different time spans, mixture model can only catch the characteristicses of a returns distribution at a fixed time span. As to grasp the whole picture of how a returns distribution evolves dynamically, we propose to model it as a Tsallis distribution, in which the statistical parameters are optimized by Genetic Algorithm (GA), since GA has the ability to reach a global optimal solution without converging into local ones [6][7][8]. The rest of this paper is organized as follows. In section 2, we simply review and summarize the evidence of returns, to show how a returns distribution evolves over time. And in section 3, we simply summarize the basic properties of Tsallis distribution and Fokker-Planck equation, and to show how to optimize the parameters by using GA. In section 4, we present its application and numerical results with real market data sets. Finally in section 5, we provide some concluding remarks.

2. Evolution of returns distribution

Returns can be calculated over different time spans, such as, one hour, one day, one week, one month, one year, and so on. Usually it tends to be different distributions over varying time spans. Each distribution has different statistical properties, such as, kurtosis, standard deviation etc. It seems that the evolving distribution is getting closer and closer to normal distribution, as the time span is getting wider and wider. But, in fact, it can be shown that most of them are not normal. Normality will be rejected by statistical tests, such as Jarque-Bera test [1][3][4][5][9][10][11].

Here returns of stock A over different time spans are shown in Table-1, where returns are calculated by $r_t = log p_t - log p_{t-1}$. Plot of the time series of its returns is shown in Fig.1. Seen from Table-1, kurtosis and standard

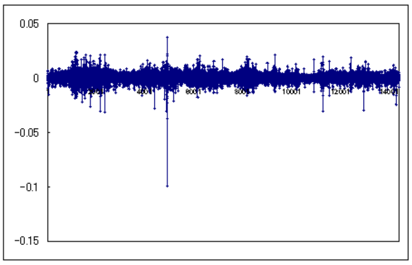

Figure 1: Plots of returns of Stock A

Stock A	Kurtosis	S.D.
1-day	35.15	0.00389
7-day	8.067	0.00984
14-day	5.844	0.01423
21-day	3.854	0.01772
28-day	3.214	0.02048
56-day	2.550	0.02896
112-day	0.839	0.04148
224-day	0.257	0.06028

Table 1: Evolution of kurtosis and standard deviation of Stock A over different time spans

deviation are getteing nearer and nearer to normal as time span evolves wider and wider. But, it never reaches normal level.

3. Tsallis distribution

3.1. Tsallis entropy and Fokker-Planck equation

Tsallis entropy is defined as follows [11][12][13]. It is straight forward that S_q will converge into a usual entropy when q takes limit to 1, namely, $S = -\int PInP$. Here, $P(x,t)$ is probability density function (p.d.f) at time t, and parameter q is independent of time t.

$$S_q = -\frac{1}{1-q}(1 - \int P(x,t)^q)dx \tag{1}$$

The following equations can be introduced as constaints. Equation (2) works as a constraint as to make $P(x,t)$ as a p.d.f in common sense. Equation (3)(4) and (5)(6) are so called q-mean, and q-variance. They are different from usual mean and variance unless $q = 1$.

$$\int P(x,t)dx = 1 \tag{2}$$

$$E(x - \bar{x}(t))_q = \int (x - \bar{x}(t))P(x,t)^q dx \tag{3}$$

$$= 0 \tag{4}$$

$$E(x - \bar{x}(t))_q^2 = \int (x - \bar{x}(t))^2 P(x,t)^q dx \tag{5}$$

$$= \sigma_q(t)^2 \tag{6}$$

By maximizing the Tsallis entropy constrained by above equation (2)-(6) for some fixed q, it yields,

$$P(x,t) = \frac{1}{Z(t)}(1 + \beta(t)(q-1)(x - \bar{x}(t))^2)^{\frac{1}{1-q}} \tag{7}$$

Where $Z(t)$ and $\beta(t)$ are Lagrange multipliers corresponding to equation (2) and (5)(6).

$$Z(t) = \frac{B(\frac{1}{2}, \frac{1}{q-1} - \frac{1}{2})}{\sqrt{(q-1)\beta(t)}} \tag{8}$$

$$\beta(t) = \frac{1}{2\sigma_q(t)^2 Z(t)^{q-1}} \tag{9}$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{10}$$

Since

$$\sigma^2(t) = E(x - \bar{x}(t))^2 \tag{11}$$

then

$$\sigma^2(t) = \begin{cases} \frac{1}{(5-3q)\beta(t)} & \text{if } q < \frac{5}{3} \\ \infty & \text{if } q \geq \frac{5}{3} \end{cases} \tag{12}$$

Where q would be less than $\frac{5}{3}$ if it has a distribution with finite variance, otherwise it has a distribution with infinite variance. However, sometimes, it is difficult to assert that returns distribution only has a finite variance from market data sets.

For the nonlinear Fokker-Planck equation

$$\frac{\partial P(x,t)^\mu}{\partial t} = -\frac{\partial}{\partial x}(F(x)P(x,t)^\mu) + \frac{D\partial^2 P(x,t)^\nu}{2\partial x^2} \tag{13}$$

It can be solved by Equation (7) when $q = 1 + \mu - \nu$. Where F(x) is supposed to be a linear drift term, namely, $F(x) = a - bx$. And here if

$$\frac{dx}{dt} = F(x) + \sqrt{DP(x,t)^{1-q}}\xi(t) \tag{14}$$

where $\xi(t)$ is a Gaussian noise, namely,

$$< \xi(t)\xi(t') >= \delta(t - t') \tag{15}$$

Where the diffusion coefficient term is $DP(x,t)^{1-q}$, and it is called as subdiffusion in the case $q < 1$, and called as superdiffusion in the case $q > 1$. It is clearly different from the normal Brownian motion, since the diffusion coefficient term is only D in the normal Brownian motion. Thus, this model can be used as to fit nonlinear diffusion process. Its application is shown in the numerical experiments in section 4 .

It can be derived from a general Ito-Langevin as follows [12][13].

$$\frac{dx}{dt} = a(x,t) + b(x,t)\xi(t) \tag{16}$$

$$\frac{d < f(x,t) >}{dt} = \int dx(a(x,t)\frac{df}{dx} + \frac{1}{2}b^2(x,t)\frac{d^2 f}{dx^2})P(x,t) \quad (17)$$

Integrating by patrs, it yields,

$$\frac{d < f(x(t)) >}{dt} = \int dx f(x)(-\frac{\partial}{\partial x}(a(x,t)P(x,t)) \quad (18)$$

$$+ \frac{\partial^2}{2\partial x^2}(b^2(x,t)P(x,t))) \quad (19)$$

It also can be written as

$$\frac{d < f(x(t)) >}{dt} = \frac{d}{dt}\int dx f(x)P(x,t) \quad (20)$$

$$= \int dx f(x)\frac{\partial P(x,t)}{\partial t} \quad (21)$$

Thus, from above equations (18)(19)(20), it yields,

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}(a(x,t)P(x,t)) + \frac{\partial^2}{2\partial x^2}(b^2(x,t)P(x,t)) \quad (22)$$

Here, if setting $a(x,t) = F(x)$, $b(x,t) = D^{\frac{1}{2}}P(x,t)^{\frac{1-q}{2}}$, $\mu = 1$, and $\nu = 2 - q$, then (13) is obtained. And, with

$$\begin{aligned}\beta(t)^{(q-3)/2} &= \beta(t_0)^{(q-3)/2} exp(b(q-3)(t-t_0)) \\ &- 2Db^{-1}(2-q)(\beta(t_0)Z^2(t_0))^{(q-1)/2} \\ &* (exp(-b(3-q)(t-t_0)) - 1)\end{aligned}$$

Equation (7) solves (13).

3.2. GA-based parameter optimization

However, it is hard to estimate parameters in Tsallis distribution, since one has to estimate q, and $Z(t)$ or $\beta(t)$ simultaneously. It becomes more complicated when there are several data sets available for several different time spans. It is necessary to consider each fitting result of each different time span. Thus, it turns out to be a multiobjective optimization problem. Usually it is not easy to get optimal solution in dealing with such a multiobjective optimization problem. Usual optimization methods probably converge into some local optimal solutions. Here we propose to get parameters optimized by using GA. So far, GA, as one of the most efficient optimization methods, which converges rather into a global solution than a local one in search of optimal solution, has been widely applied in many research fields ranging from scientific researches to social studies [5][6][7][8].

Here, suppose that we have several data sets for different tiem spans, then we can get several likelihood functions for the data sets, say, $L_0, L_1, ..L_{m-1}$ which share the same parameter q, with different parameters $\beta(t_0)$, $\beta(t_1)$, $\ldots$, $\beta(t_{m-1})$. Let $V = \Sigma_{i=0}^{m-1} L_i$, we consider that, the optimal solution is a set of q, $\beta(t_0)$, $\beta(t_1)$, $\ldots$, $\beta(t_{m-1})$ which makes V reach the maximum.

Our GA scheme is designed as follows.

Step 1, to generate random numbers as individuals of the first generation with certain population. Here, each individual represents a set of parameters in P(x,t) (equation (7)), namely, q, and $\beta(t)$ or $Z(t)$.

Step 2, to evaluate the fitness of each individual based on predetermined fitness function, then to sort all individuals of the generation according to their fitness values.

Step 3, to select two individuals with higher fitness values from the generation at a certain probability. The selection strategy has a great deal pf variations, Roulette strategy is adopted in our applications.

Step 4. to apply genetic operations, namely, crossover operation, and mutation operation to two selected individuals to reproduce their offsprings and put them into the pool of next generation.

Here, a crossover operation means to randomly decide crossover positions on the two selected individuals at first, then to exchange parts of two individuals each other. Basically, there are two methods to do this, one is one-point crossover, the other is multipoints crossover. The later one is applied to our application. A mutation operation means to randomly decide mutation positions under a certain probability, and then to change those position values of a selected individual. It also has two ways to do this. One is one-point mutation, the other is multipoints mutation. The later one is adopted in our application.

Step 5, to reevaluate the fitness of each individual of the new generation, to see if the results meet the terminal conditions, such as repeating times, or error range etc, if it does, then GA terminates, else it goes back to Step 3.

And fitness function for evaluating jth individual is defined as

$$Fitness_j = \frac{1}{(n-1)}\frac{V_j}{\sum_j V_j} \quad (23)$$

where V_j is a sum of likelihood values corresponding to L_is.

4. Applications

In this section, we apply our proposed method to real market data set A. Data set A is the daily stock price of stock A, from Jan, 2, 1980 ~ Mar, 29, 2007. We fit Tsillas distribution with 1-day, 14-day, 28-day returns. Our GA's parameters are set as follows, population size = 200, Crossover and mutation probabilities are 0.42 and 0.31 respectively. And $q \in [0.001, 50]$, $\beta(t) \in [0.001, 50]$. Furthermore, we employ elite-keeping policy in GA. An elite-keeping policy is to copy an individual with higher or highest fitness into next generation automatically. We repeat GA for times and get the same global optimal solution, where, estimated q and $\beta(t_0)$ are 2.23, and 1.68 respectively. It is a superdiffusion process since $q = 2.23 > 1$.

We show the results of their estimated distributions compared with their empirical distributions in Fig.2, 3, 4. Seen from these figures, the numerical results that the estimated

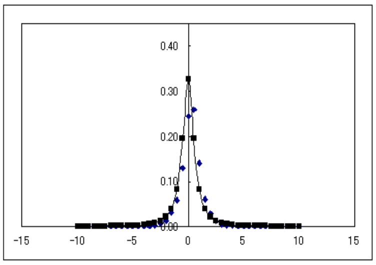

Figure 2: Plot of p.d.f of day-1

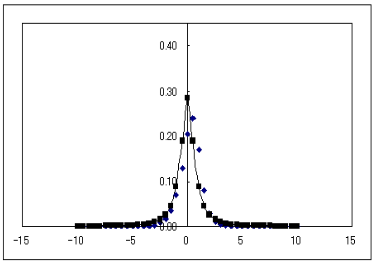

Figure 3: Plot of p.d.f of day-14

$P(x, t)$ dynamically traces the evolution of a returns distribution over different time spans.

5. Concluding remarks

In this study, we apply GA-based parameter optimization of Tsallis distribution to real market data, and we find it works well by exactly catching the evolution behavior of the probability density function of the data sets over different time spans. It provides us a whole picture of how the returns distribution evolves over varying time intervals, not only at one fixed time span. It is important for further researches on the whole time axis, such as evaluating the Value at Risk over the different time spans.

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References

[1] Tsay, R. S., Analysis of financial time series, John Wiley & Sons, 2002.

[2] Mantegana, R., N., Stanley, H., E., An introduction to econophysics: correlations and complexity in finance, Cambridge University Press, 1999.

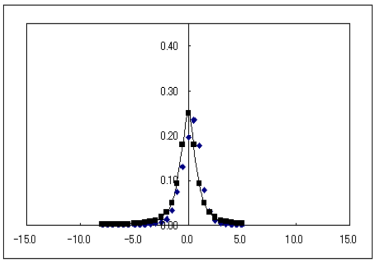

Figure 4: Plot of p.d.f of day-28

[3] Campbell, J. Y., Lo, A. W., The econometrics of financial markets, Princeton Press, 1997.

[4] Gabaix, X., et al., ”A theory of power law distribution in financial market fluctuations”, *Nature*, 423, 2003, pp267-270.

[5] Tan K., Tokinaga S., 2006. ”Identifying returns distribution by using mixture distribution optimized by Genetic alogrithm”, Proceeding of NOLTA2006, pp.119-122, 2006.

[6] Goldberg, D., Genetic algorithm: in search, optimization, and machine learning, Addison-Wesley Press, 1989.

[7] Tan K., Tokinaga S., ”Optimization of fuzzy inference rules by using the genetic algorithm and its application to bond rating”, *Journal of Operations Research Japan*, vol.42, 3, pp.302-315. 1999.

[8] Tan K., Tokinaga S., ”The design of multistage fuzzy inference system with small number of rules based upon the optimization of rules by using GA”. Trans. IEEE, E82-A, 9, pp.1865-1873, 1999b.

[9] Tan, K., ”Evaluation of VaR in the Japanese financial market”, *Journal of the society for studies on industrial economies*, vol.43, 2, pp.449-467, 2002a.

[10] Tan, K., ”Fractality in financial market”, *Journal of the society for studies on industrial economies*, vol.43, 3, pp.553-566, 2002b

[11] Tsallis, C., Bukman, D.J., 1996, Anomalous diffusion in the presence of external forces: Exact time-dependent solutions and entropy, Physical Review E 54, pp.2197.

[12] Micheal, F., Johnson, M., D., ”Financial market dynamics”, *Physica A: Statistical mechanics and its applications*, vol.320, 15 March 2003.

[13] Heyde, C., C., Leonenko, N., N., ”Student processes”, *Appl. Prob*, Vol.37, pp.342-365, 2005.