Performance Consensus Problem of Multi-agent Systems with Multiple State Variables

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Abstract—This paper addresses a discrete-time consensus problem of multi-agent systems. In our model, each agent has multiple state variables instead of a scalar one. A performance value of each agent is evaluated by a nonlinear performance function based on its state. We consider conditions for agents to achieve consensus on the performance value using the algebraic graph theory and the mean value theorem. The proposed method is applied to multi-resource allocation for a group of agents to achieve performance equalization.

1. Introduction

Over the past several years, coordinated behaviors of multi-agent systems have been much attention to in engineering and scientific fields such as decision making in animal groups, sensors networks, and transportation systems ([1], [2]). While there are many researches for the multi-agent systems, recent studies on a consensus problem have revealed significant mechanisms for a group of agents to reach consensus, that is, to have a common decision or to show the same response as a group ([3], [4], [5], [6], [7]). For instance, D. P. Bertsekas and J. N. Tsitsiklis investigated the distributed asynchronous consensus algorithm in the context of parallel computation [3]. W. Ren and R. W. Beard also studied the information consensus problem of multiple agents using discrete-time and continuous-time consensus algorithms [6]. On the other hand, L. Xiao and S. Boyd provided the optimal center-free algorithm for distributed resource allocation with convex cost functions [7].

Recently, we addressed the consensus problem of multi-agent systems with nonlinear performance functions [8]. In the consensus problem, a state of each agent is usually assumed to be a scalar variable. The purpose of this paper is to extend our previous work to the more general case where each agent has multiple state variables rather than a scalar state. The performance value of each agent is evaluated by a nonlinear performance function according to its current state. Throughout this paper, we say that a group of agents achieved consensus when all agents have the same performance value. We show conditions for the consensus among agents using the algebraic graph theory and the mean value theorem. As an application of the proposed method, we also consider a fair multi-resource allocation problem, where several resources are dynamically allocated to each agent so that performances of all agents are equal.

This paper is organized as follows. We begin by reviewing the graph theory and the matrix theory in Section 2. In Section 3, we explain a mathematical model for locally interacting multi-agent systems. Section 4 considers conditions for the consensus among agents on their performance values. Section 5 focuses on an application of our theory to a fair multi-resource allocation problem. Section 6 shows a numerical example for the resource allocation problem discussed in Section 5. Finally, we state concluding remarks in Section 7.

2. Preliminaries

We briefly review the fundamental facts of the graph theory and the matrix theory [3], [6], [9], [10].

2.1. Graph

The topology of communication networks among n agents is modeled as a time-varying weighted digraph $G(V, E)$ with a node set $V = \{v_i \mid i \in \mathcal{I}\}$ and an edge set $E \subseteq V \times V$, where $\mathcal{I} = \{1, 2, \ldots, n\}$. Agents are labeled by 1 through n and each node v_i in the digraph represents the individual agent i. Each directed edge (v_i, v_j) $(\in E)$ indicates the unidirectional communication from agent i to agent j. A directed tree is a digraph whose nodes except the root have exactly one parent. A spanning tree of a digraph is a directed tree formed by unidirectional graph edges that connect all the nodes of the tree. A directed graph is said to have a spanning tree if the graph contains a spanning tree as a subgraph.

Let $G_i(V, E_i)$ $(i = 1, 2, \ldots, M)$ be a possible interaction graph with a common node set V. The union of the graphs $G = \bigcup_{i=1}^{M} G_i$ is the digraph with the common node set V and the union of the edge sets $\bigcup_{i=1}^{M} E_i$.

2.2. Matrix

A vector p $(\in \mathbb{R}^n)$ is said to be positive or nonnegative, denoted by $p > 0$ or $p \geq 0$, if all components of p are positive or nonnegative, respectively. For any two vectors p $(\in \mathbb{R}^n)$ and q $(\in \mathbb{R}^n)$, the notations $p > q$ or $p \geq q$ stand for $p - q > 0$ or $p - q \geq 0$, respectively. Similarly, a matrix $P = [p_{ij}]$ $(\in \mathbb{R}^{n \times n})$ is said to be nonnegative, denoted by

$P \geq 0$, if $p_{ij} \geq 0$ for all i, j ($\in \mathcal{I}$). A (row) stochastic matrix P is a nonnegative matrix which satisfies $p_{ij} \geq 0$ for all i, j ($\in \mathcal{I}$) and $\sum_{j=1}^{n} p_{ij} = 1$.

3. Systems Model

We consider a system with n agents each of which has a performance value and m independent information as its state. Let $x_i[k] = [x_{i_1}[k] \, x_{i_2}[k] \, \cdots \, x_{i_m}[k]]^T$ ($\in \mathbb{R}^m$) be the state vector of agent i ($i \in \mathcal{I}$) at time k ($k = 0, 1, 2, \ldots$). The performance value corresponding to the state vector is denoted by $y_i[k]$ ($\in \mathbb{R}$) and characterized by the performance function $f_i(x_i) = f_i(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$. We assume that each performance function $f_i(x_i)$ is strictly increasing and of class C^1.

The mathematical model of the system is described by

$$x_i[k+1] = x_i[k] + \beta_i[k] \left\{ \sum_{j=1, j \neq i}^{n} \alpha_{ij}[k] g_{ij}[k] (y_j[k] - y_i[k]) \right\}, \quad (1)$$

$$y_i[k] = f_i(x_i[k]), \quad (2)$$

where $\alpha_{ij}[k]$ ($\in \mathbb{R}$) is the positive time-varying weight on the directed edge (v_j, v_i) ($\in E$), $\beta_i[k] = [\beta_{i_1}[k] \, \beta_{i_2}[k] \, \cdots \, \beta_{i_m}[k]]^T > 0$ ($\in \mathbb{R}^m$) and $g_{ij}[k]$ is a time-varying Boolean variable which represents the presence of information from agent j to agent i:

$$g_{ij}[k] = \begin{cases} 1, & \text{if agent } i \text{ receives information from} \\ & \text{agent } j \text{ at time } k, \\ 0, & \text{otherwise,} \end{cases}$$

and $g_{ii}[k] = 1$ for any k. We assume that $\alpha_{ij}[k]$ and $\beta_{i_\ell}[k]$ are uniformly lower and upper bounded, that is,

$$\alpha^{\text{inf}} \leq \alpha_{ij}[k] \leq \alpha^{\text{sup}}, \quad \beta^{\text{inf}} \leq \beta_{i_\ell}[k] \leq \beta^{\text{sup}}.$$

We say that a group of agents achieves consensus if Eq. (3) holds for any i, j ($\in \mathcal{I}$) and for any initial state.

$$\left| y_i[k] - y_j[k] \right| \to 0, \quad \text{as } k \to \infty. \quad (3)$$

4. Global Consensus

In the subsequent discussion, we assume that any state variable $x_{i_\ell}[k]$ ($\ell = 1, 2, \ldots, m$) is in the interval $[x_{i_\ell}^{\text{inf}}, x_{i_\ell}^{\text{sup}}]$.
Define $\gamma_{i_\ell}[k]$ as

$$\gamma_{i_\ell}[k] = \begin{cases} \left. \dfrac{\partial}{\partial x_{i_\ell}} f_i(x_i) \right|_{x_i = x_i^c[k]}, & \text{if } x_{i_\ell}[k+1] \neq x_{i_\ell}[k], \\ \left. \dfrac{\partial}{\partial x_{i_\ell}} f_i(x_i) \right|_{x_i = x_i[k]}, & \text{if } x_{i_\ell}[k+1] = x_{i_\ell}[k], \end{cases} \quad (4)$$

where $x_i^c[k] = [x_{i_1}^c[k] \, x_{i_2}^c[k] \, \cdots \, x_{i_m}^c[k]]^T$ ($\in \mathbb{R}^m$) and $\min\{x_{i_\ell}[k], x_{i_\ell}[k+1]\} < x_{i_\ell}^c[k] < \max\{x_{i_\ell}[k], x_{i_\ell}[k+1]\}$.

Applying the mean value theorem to Eq. (2), we have

$$y_i[k+1] = y_i[k] + \left\{ \sum_{\ell=1}^{m} \gamma_{i_\ell}[k] (x_{i_\ell}[k+1] - x_{i_\ell}[k]) \right\}. \quad (5)$$

Then, we can rewrite Eqs. (1) and (2) as follows:

$$x[k+1] = x[k] + B[k]A[k]y[k], \quad (6)$$
$$y[k+1] = y[k] + C[k](x[k+1] - x[k]), \quad (7)$$

where

$$x[k] = \left[x_1^T[k] \, x_2^T[k] \, \cdots \, x_n^T[k] \right]^T \in \mathbb{R}^{mn},$$

$$y[k] = \left[y_1[k] \, y_2[k] \, \cdots \, y_n[k] \right]^T \in \mathbb{R}^n,$$

$$A[k] = \begin{bmatrix} -\sum_{j=2}^{n} \alpha_{1j}[k]g_{1j}[k] & \cdots & \alpha_{1n}[k]g_{1n}[k] \\ \vdots & & \vdots \\ -\sum_{j=2}^{n} \alpha_{1j}[k]g_{1j}[k] & \cdots & \alpha_{1n}[k]g_{1n}[k] \\ \vdots & \ddots & \vdots \\ \alpha_{n1}[k]g_{n1}[k] & \cdots & -\sum_{j=1}^{n-1} \alpha_{nj}[k]g_{nj}[k] \\ \vdots & & \vdots \\ \alpha_{n1}[k]g_{n1}[k] & \cdots & -\sum_{j=1}^{n-1} \alpha_{nj}[k]g_{nj}[k] \end{bmatrix} \in \mathbb{R}^{mn \times n},$$

$$B[k] = \text{diag} \left[\beta_{1_1}[k] \, \cdots \, \beta_{1_m}[k] \, \cdots \, \beta_{n_1}[k] \, \cdots \, \beta_{n_m}[k] \right] \in \mathbb{R}^{mn \times mn},$$

$$C[k] = \begin{bmatrix} \gamma_{1_1}[k] & \cdots & \gamma_{1_m}[k] & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & \gamma_{n_1}[k] & \cdots & \gamma_{n_m}[k] \end{bmatrix} \in \mathbb{R}^{n \times mn}.$$

From Eqs. (6) and (7), we have

$$\begin{aligned} y[k+1] &= y[k] + C[k]B[k]A[k]y[k] \\ &= (I_n + C[k]B[k]A[k])y[k] \\ &= W[k]y[k], \quad (8) \end{aligned}$$

where I_n is an $n \times n$ identity matrix and $W[k] = I_n + C[k]B[k]A[k]$ ($\in \mathbb{R}^{n \times n}$).

Let $d_i^{\text{in}}[k]$ be the number of incoming edges to node v_i ($i \in \mathcal{I}$). Note that $x_i[k+1] = x_i[k]$ for any $\beta_i[k]$ if $d_i^{\text{in}}[k] = 0$. Thus, we assume that all agents receive information from at least one agent, that is, $d_i^{\text{in}}[k] \neq 0$ for all i and k.

Lemma 1 *Let* $\gamma_{i_\ell}^{\text{sup}} = \sup_k \gamma_{i_\ell}[k] = \sup_{x_i} \dfrac{\partial}{\partial x_{i_\ell}} f_i(x_i)$. *The matrix* $W[k]$ *is a stochastic matrix with positive diagonal entries if any edge weight* $\alpha_{ij}[k]$ *satisfies*

$$0 < \alpha_{ij}[k] < \cfrac{1}{\left(\displaystyle\sum_{\ell=1}^{m} \gamma_{i_\ell}^{\mathrm{sup}} \beta_{i_\ell}[k]\right) d_i^{in}[k]}. \qquad (9)$$

Proof: The diagonal and non-diagonal entries of $W[k] = \left[w_{ij}[k]\right]$ in Eq. (8) are given by

$$w_{ii}[k] = 1 - \left(\sum_{\ell=1}^{m} \gamma_{i_\ell}[k]\beta_{i_\ell}[k]\right)\left(\sum_{j=1,j\neq i}^{n} \alpha_{ij}[k]g_{ij}[k]\right), (10)$$

$$w_{ij}[k] = \left(\sum_{\ell=1}^{m} \gamma_{i_\ell}[k]\beta_{i_\ell}[k]\right)\alpha_{ij}[k]g_{ij}[k], \quad \text{if } i \neq j. \quad (11)$$

Note that $\gamma_{i_\ell}[k]$ is positive for all k since the performance function $f_i(x_i)$ is strictly increasing. Thus, all non-diagonal entries of $W[k]$ are positive.

Since $d_i^{\mathrm{in}}[k] \neq 0$, from Eq. (9), we have

$$0 < \alpha_{ij}[k] < \cfrac{1}{\left(\displaystyle\sum_{\ell=1}^{m} \gamma_{i_\ell}[k]\beta_{i_\ell}[k]\right) d_i^{\mathrm{in}}[k]}. \qquad (12)$$

Let $\alpha_i^{\max}[k] = \max\limits_{j\in\mathcal{I}, j\neq i}\left\{\alpha_{ij}[k] \mid g_{ij}[k] = 1\right\}$. Then, we have

$$\begin{aligned}
w_{ii}[k] &= 1 - \left(\sum_{\ell=1}^{m} \gamma_{i_\ell}[k]\beta_{i_\ell}[k]\right)\left(\sum_{j=1,j\neq i}^{n} \alpha_{ij}[k]g_{ij}[k]\right) \\
&\geq 1 - \left(\sum_{\ell=1}^{m} \gamma_{i_\ell}^{\mathrm{sup}} \beta_{i_\ell}[k]\right)\alpha_i^{\max}[k]d_i^{\mathrm{in}}[k] > 0. \quad (13)
\end{aligned}$$

On the other hand, from Eqs. (10) and (11), we have

$$w_{ii}[k] + \sum_{j=1,j\neq i}^{n} w_{ij}[k] = 1, \quad \text{for any } i, k. \qquad (14)$$

From Eqs. (13) and (14), we conclude that $W[k]$ is a stochastic matrix with positive diagonal entries if Eq. (9) holds.

□

Theorem 1 *Suppose that $d_i^{\mathrm{in}}[k]$ is positive for all i and k. Let $x_i[0] > 0$ ($i \in \mathcal{I}$). All performance values $y_i[k]$ starting from any initial state converges to an equilibrium point using Eqs. (1) and (2) if any edge weight $\alpha_{ij}[k]$ satisfies*

$$0 < \alpha_{ij}[k] < \cfrac{1}{\left(\displaystyle\sum_{\ell=1}^{m} \gamma_{i_\ell}^{\mathrm{sup}} \beta_{i_\ell}[k]\right) d_i^{in}[k]}, \qquad (15)$$

and there exists an infinite sequence $0 = k_0 < k_1 < k_2 < \cdots$ such that

(i) the time interval $[0,\infty)$ is divided into non-overlapping subintervals $[k_0, k_1) \cup [k_1, k_2) \cup \cdots$,

(ii) each subinterval $[k_s, k_{s+1})$ ($s = 0, 1, 2, \ldots$) is uniformly bounded and the union of the interaction graphs across each subinterval has a spanning tree.

Proof: From the assumption that $\alpha_{ij}[k]$, $\beta_{i_\ell}[k]$ and $\gamma_{i_\ell}[k]$ are uniformly lower and upper bounded, all nonzero entries of the $W[k]$ in Eq. (8) are also uniformly lower and upper bounded. Moreover, from Lemma 1, $W[k]$ is a stochastic matrix with positive diagonal entries. Thus, by following the similar argument of Theorem 3.2 in [10], we have

$$y[k] \longrightarrow \mu\mathbf{1}, \quad \text{as} \quad k \longrightarrow \infty, \qquad (16)$$

where μ ($\in \mathbb{R}$) is a constant value and $\mathbf{1} = [1\ 1\ \cdots\ 1]^T$ ($\in \mathbb{R}^n$).

□

5. Application to Fair Resource Allocation

We next consider fair multi-resource allocation to equalize the performance of agents under the following resource constrains:

$$\sum_{i=1}^{n} x_{i_\ell}[k] = R_\ell, \quad \text{for all } k, \qquad (17)$$

where $\ell = 1, 2, \ldots, m$.

In this section, to incorporate the resource constraints into our model, we use an undirected graph for the communication networks of agents, that is, $\alpha_{ij}[k] = \alpha_{ji}[k]$ and $g_{ij}[k] = g_{ji}[k]$ for any i, j and k. We also assume that the performance functions $f_i : \mathbb{R}^m \mapsto \mathbb{R}$ satisfy

- strictly increasing and of class C^1,

- $f_i(x_i) = 0 \Leftrightarrow x_{i_\ell} = 0$ for some ℓ,

- $f_i(x_i) > 0 \Leftrightarrow x_i > 0$.

Proposition 1 *Let $x_i[0] > 0$. The state vector $x_i[k]$ is positive for all k if any edge weight $\alpha_{ij}[k]$ satisfies*

$$0 < \alpha_{ij}[k] < \cfrac{1}{\left(\displaystyle\sum_{\ell=1}^{m} \gamma_{i_\ell}^{\mathrm{sup}} \beta_{i_\ell}[k]\right) d_i^{in}[k]}, \qquad (18)$$

Proof: Suppose that $x_i[k]$ is positive for some k. From the assumptions of the performance functions, we know that the performance value $y_i[k]$ is positive. If Eq. (18) holds, $W[k]$ is a stochastic matrix with positive diagonal entries, and hence $y_i[k + 1]$ is also positive. This immediately follows that $x_i[k + 1]$ is positive.

Considering the preceding argument and the initial condition $x_i[k] > 0$, we conclude that the state vector $x_i[k]$ is positive for all k.

□

6. Simulation Results

We consider the resource allocation problem discussed in Section 5 with 5 agents. Each agent has a performance value $y_i[k]$ and two independent information $x_{i_1}[k]$ and

$x_{i_2}[k]$ ($i \in \mathcal{I} = \{1, 2, \ldots, 5\}$). We give the initial values of the states of agents as shown in Table 1 under the following resource constraints:

$$\sum_{i=1}^{5} x_{i_1}[k] = 1.0, \quad \sum_{i=1}^{5} x_{i_2}[k] = 1.5, \quad \text{for all } k.$$

The performance functions are given as follows:

$$f_1(x_{1_1}, x_{1_2}) = \frac{1}{2}x_{1_1}^2 x_{1_2} + x_{1_1} x_{1_2},$$

$$f_2(x_{2_1}, x_{2_2}) = \frac{1}{2}x_{2_1} x_{2_2} + \frac{1}{4}x_{2_1} x_{2_2}^2,$$

$$f_3(x_{3_1}, x_{3_2}) = \frac{3}{2}x_{3_1} x_{3_2},$$

$$f_4(x_{4_1}, x_{4_2}) = \frac{1}{4}x_{4_1} x_{4_2}^2,$$

$$f_5(x_{5_1}, x_{5_2}) = \frac{1}{4}x_{5_1}^2 x_{5_2} + \frac{1}{2}x_{5_1} x_{5_2},$$

where $0 \le x_{i_1} \le 1$ and $0 \le x_{i_2} \le 1.5$ ($i \in \mathcal{I}$). Then, we have $\gamma_{i_\ell}^{\sup}$ as shown in Table 2 ($\ell = 1, 2$). We also set the weight $\alpha_{ij}[k]$ ($i, j \in \mathcal{I}$) and $\beta_{i_\ell}[k]$ as follows:

$$\alpha_{ij}[k] = 0.9 \times$$

$$\min \left\{ \frac{1}{\left(\sum_{\ell=1}^{2} \gamma_{i_\ell}^{\sup} \beta_{i_\ell}[k] \right) d_i^{\text{in}}[k]}, \frac{1}{\left(\sum_{\ell=1}^{2} \gamma_{j_\ell}^{\sup} \beta_{j_\ell}[k] \right) d_j^{\text{in}}[k]} \right\},$$

$$\beta_{i_\ell}[k] = 1, \quad \text{for all } i, \ell \text{ and } k.$$

Table 1: Initial values of the states of agents.

$x_{1_1}[0]$	$x_{2_1}[0]$	$x_{3_1}[0]$	$x_{4_1}[0]$	$x_{5_1}[0]$
0.25	0.23	0.05	0.16	0.31

$x_{1_2}[0]$	$x_{2_2}[0]$	$x_{3_2}[0]$	$x_{4_2}[0]$	$x_{5_2}[0]$
0.32	0.26	0.25	0.36	0.31

Table 2: Variables $\gamma_{i_\ell}^{\sup}$ under the conditions $0 \le x_{i_1} \le 1$ and $0 \le x_{i_2} \le 1.5$ ($i \in \mathcal{I}$).

$\gamma_{1_1}^{\sup}$	$\gamma_{2_1}^{\sup}$	$\gamma_{3_1}^{\sup}$	$\gamma_{4_1}^{\sup}$	$\gamma_{5_1}^{\sup}$
3	$\frac{21}{16}$	$\frac{9}{4}$	$\frac{9}{16}$	$\frac{3}{2}$

$\gamma_{1_2}^{\sup}$	$\gamma_{2_2}^{\sup}$	$\gamma_{3_2}^{\sup}$	$\gamma_{4_2}^{\sup}$	$\gamma_{5_2}^{\sup}$
$\frac{3}{2}$	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{3}{4}$

Figure 1 illustrates the convergence process of the performance values to equilibrium. This result shows that a group of agents achieved consensus on their performance values.

7. Conclusions

We have considered a discrete-time consensus problem of locally interacting multi-agent systems. The main contribution of this paper is to investigate the case where each
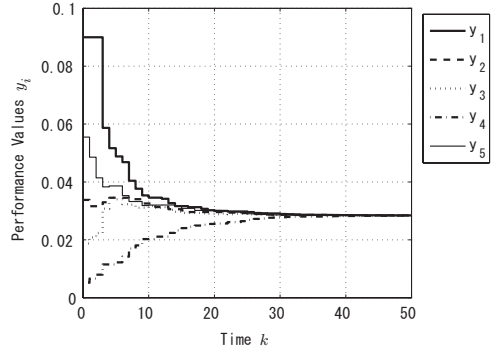

Figure 1: The consensus process of performance values $y_i[k]$ with 5 agents ($i \in \mathcal{I} = \{1, 2, \ldots, 5\}$).

agent has multiple state variables instead of a scalar one. We derived sufficient conditions for the consensus on the performance value using the algebraic graph theory and the mean value theorem. We also showed that the proposed method was applicable to a fair multi-resource allocation problem. As future work, we will need the further discussion on the broader class of problems including time-delays and optimizations.

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