2007 International Symposium on Nonlinear Theory and its Applications
NOLTA'07, Vancouver, Canada, September 16-19, 2007

Replicator Dynamics with Dynamic Payoff Reallocation Based on the Government's Payoff

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Abstract—In a population which consists of a large number of players interacting with each other, the payoff of each player often conflicts with the total payoff of the population which he/she belongs to. In such a situation, the "government" which has the comprehensive perspective is introduced for governing the population. When the government collects and reallocates players' payoffs for governing the population, the evolutions of population states are modeled by replicator-mutator dynamics. In this paper, we propose a model which describes the evolution of the government's reallocation strategy and investigate stability of its equilibrium points.

1. Introduction

A problem called social dilemma occurs when the purpose of each person conflicts with the total purpose of the community which he/she belongs to [1, 2]. Evolutionary game theory has been used as a powerful mathematical framework to analyze such a social problem [3, 4]. Especially, when the government collects and reallocates players' payoffs for governing the population, the evolutions of population states are modeled by replicator-mutator dynamics [5].

The social problem is the conflict between the payoff of each player and the total payoff of the population. Therefore, such a problem is unsolvable by personal effort of each player. The "government" which has the comprehensive perspective is required for governing the population. In the real world, it corresponds to the rulers such as the government of countries or cities, and executives of organizations or companies. The government is willing to lead the population state to a desirable state by intervening in the population. In replicator dynamics with reallocation of payoffs [5], we consider that players interact with each other in their own population and the government intervenes in the interactions by collecting and reallocating payoffs. In this model, the government's action is the rate of collecting payoffs from players and the rate is independent of the population state. However, in the case that the government can change the rate depending on the state as its policy, it can be modeled as a player.

In this paper, we define the government's payoff as a sum of benefits which depend on the current state of the population and a cost of the government's intervention. Moreover, we propose a model which describes the evolution of its payoff reallocation strategy, and investigate stability of its equilibrium points in the case that players have two strategies and the target state is on the boundary.

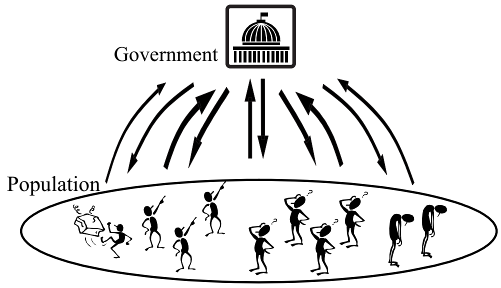


Figure 1: Collection and reallocation of payoffs.

2. Intervention by Collection and Reallocation of Payoffs

We consider a population which consists of a large number of players and the "government" which intervenes in interactions between players who belong to the population. The intervention can be modeled as a strategy of the government. Suppose that the government changes its strategy depending on the population's state. In this paper, as the government's intervention, we deal with the collections and reallocations of players' payoffs. Figure 1 shows an illustration of the collection and the reallocation of payoffs by the government.

Let P be the population of players. Suppose that $\Phi_p = \{1, 2, \cdots, m_p\}$ be a set of pure strategies of P, and S_p be a set of population states of P. A population state $s_p = (s_p^1, s_p^2, \cdots, s_p^{m_p})^T \in S_p$ is a distribution of strategies in the population P, where s_p^i is the proportion of players with a pure strategy $i \in \Phi_p$. Let $r_p^i : S_p \to \mathbb{R}$ be the payoff function for the players of P with the pure strategy $i \in \Phi_p$ and $\bar{r}_p(s_p)$ be the average payoff, i.e., $\bar{r}_p(s_p) = \sum_{i\in\Phi_p} s_p^i r_p^i(s_p)$. We assume that $r_p^i(s_p) \geq 0$ and $\bar{r}_p(s_p) > 0$.

Suppose that q^{ji} is a proportion of payoffs which is collected from players with $j \in \Phi_p$ and reallocated to players with $i \in \Phi_p$ to the total payoff of player with $j \in \Phi_p$. We call the matrix $Q = (q^{ji})$ a *reallocation matrix*. Using the above definitions, replicator dynamics with collections and reallocations of payoffs is given as follows [5]:

$$\dot{s}_p^i = \sum_{j\in\Phi_p} s_p^j r_p^j(s_p) q^{ji} - s_p^i \bar{r}_p(s_p). \tag{1}$$

Equation (1) is known as replicator-mutator dynamics [6].

Suppose that $\Phi_g = \{1, 2\}$ and S_g are sets of pure and mixed strategies of the government, respectively. We call the first strategy *complete intervention* and the second strategy *non-intervention*. In this situation, a strategy $s_g = (\alpha, 1-\alpha) \in S_g$ defines a mixed strategy between those two strategies, where α is called an *intervention rate* and assumed to satisfy $\alpha \in [0, 1]$. For the target population state $s_p^* \in S_p$, we define the reallocation matrix Q as follows:

$$Q = (1-\alpha) I_{m_p} + \alpha X^*, \tag{2}$$

where I_l is the l-dimensional unit matrix and $X^* = [s_p^* \cdots s_p^*]^T$. In this case, Eq. (1) is rewritten as follows:

$$\dot{s}_p^i = (1-\alpha) s_p^i \left\{ r_p^i(s_p) - \bar{r}_p(s_p) \right\} + \alpha \left(s_p^{i*} - s_p^i \right) \bar{r}_p(s_p). \tag{3}$$

When $\alpha = 0$ holds (non-intervention), no payoffs of players are collected and reallocated. Eq. (3) is rewritten as

$$\dot{s}_p^i = s_p^i \left\{ r_p^i(s_p) - \bar{r}_p(s_p) \right\}. \tag{4}$$

On the other hand, when $\alpha = 1$ holds (complete intervention), the government collects all payoffs of all players and reallocates them depending on the target state s_p^*. In this case, Eq. (3) is rewritten as

$$\dot{s}_p^i = \left(s_p^{i*} - s_p^i \right) \bar{r}_p(s_p). \tag{5}$$

Since $\bar{r}_p(s_p) > 0$, the target state s_p^* is a globally asymptotically stable equilibrium point of Eq. (3).

The following proposition about Eqs. (3) and (4) has been proved [7].

Proposition 1 If the target state $s_p^* \in S_p$ is an equilibrium point of Eq. (4), then it is an equilibrium point of Eq. (3) for any $\alpha \in [0, 1]$. On the other hand, if s_p^* is not an equilibrium point of Eq. (4), then there does not exist $\alpha \in [0, 1)$ such that it is not an equilibrium point of Eq. (3).

From Proposition 1, if s_p^* is not an equilibrium point of Eq. (3), then the government has to adopt the strategy *complete intervention* for leading the population state of P to the target state s_p^*. Therefore, we consider the case that the target state s_p^* is an equilibrium point of Eq. (4) in this paper.

3. Dynamic Intervention Rate

Let $r_g^i : S_p \times S_g \to \mathbb{R}$ be the payoff function for the government with the pure strategy $i \in \Phi_g$ and $\bar{r}_g(s_p, s_g)$ be the government's current payoff, i.e., $\bar{r}_g(s_p, s_g) = \alpha r_g^1(s_p, s_g) + (1-\alpha) r_g^2(s_p, s_g)$. In this paper, we suppose that the government increases the intervention rate α in proportion to differences between the payoffs of the complete intervention strategy r_g^1 and the current payoffs $\bar{r}_g$ which the government earns. Such a rule of changing the intervention rate can also be modeled by replicator dynamics. Since the government has two strategies complete intervention and non-intervention, the replicator dynamics is formulated as follows:

$$\dot{\alpha} = \alpha(1-\alpha) \left\{ r_g^1(s_p, s_g) - r_g^2(s_p, s_g) \right\}. \tag{6}$$

Using payoff matrices A, B, and C, we define the players' and the government's payoff as $r_p^i(s_p) = e_{m_p}^{iT} A s_p$ and $r_g^i(s_p, s_g) = e_2^{iT} B s_p + e_2^{iT} C s_g$, respectively, where e_l^i is the l-dimensional unit vector such that the ith element equals 1.

The matrices B and C are the payoffs which depend on the current population state of P and the current intervention rate, respectively. We consider that the matrix B is the government's benefit depending on the current population state of P and the matrix C is a cost of the government's intervention depending on the current intervention rate. Assuming that the non-intervention strategy makes no benefits and costs, we define the matrices A, B, and C as follows:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1m_p} \\ \vdots & \ddots & \vdots \\ a_{m_p 1} & \cdots & a_{m_p m_p} \end{bmatrix},$$
$$B = \begin{bmatrix} b_1 & \cdots & b_{m_p} \\ 0 & \cdots & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where $a_{ij} \geq 0$ for all i and $j \in \Phi_p$, $b_i \geq 0$ for all $i \in \Phi_p$, and $c_1 < 0$.

4. Two Strategy Game: Boundary Target Point

Suppose that players of the population P have two strategies and the target point is $s_p^* = (1, 0)^T$. Since $s_p^1 + s_p^2 = 1$, we have

$$\dot{s}_p^1 = \left(1 - s_p^1\right) \left\{ d_1 \left(s_p^1\right)^2 + d_2 s_p^1 + \alpha a_{22} \right\}, \tag{7}$$
$$\dot{\alpha} = -c_1 \alpha (1-\alpha) \left\{ (\beta_1 - \beta_2) s_p^1 + \beta_2 - \alpha \right\}, \tag{8}$$

where

$$d_1 = a_{11} - a_{21} - a_{12} + a_{22}, \tag{9}$$
$$d_2 = (a_{12} - a_{22}) + \alpha(a_{21} - a_{22}), \tag{10}$$
$$\beta_1 = -\frac{b_1}{c_1}, \quad \beta_2 = -\frac{b_2}{c_1}. \tag{11}$$

Since β_1 and β_2 are the ratios of the government's benefits b_1 and b_2 to the intervention cost c_1, we consider them as cost-efficiencies of the government's interventions to players' strategies 1 and 2, respectively.

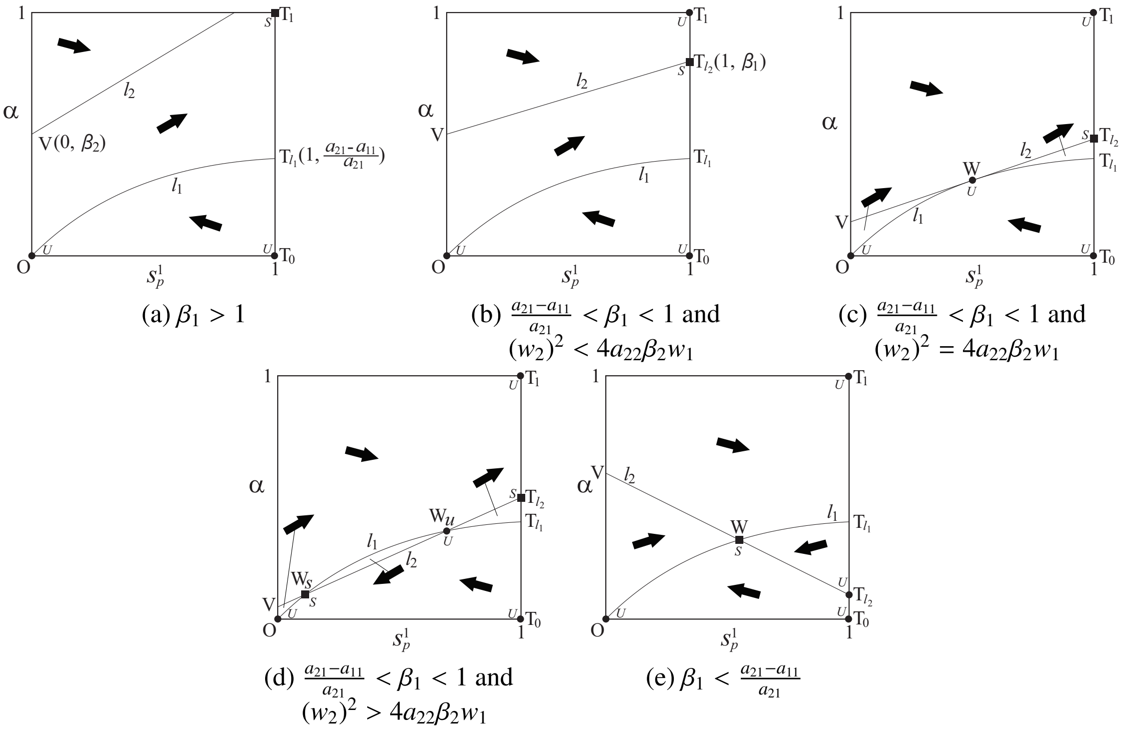


Figure 2: Schematic of vector fields and equilibrium points of the two strategy games. Black squares marked by S are stable equilibrium points and black points marked by U are unstable equilibrium points.

We have $\dot{s}_p^1 = 0$ if $s_p^1 = 1$ or $d_1\left(s_p^1\right)^2 + d_2 s_p^1 + \alpha a_{22} = 0$. On the other hand, we have $\dot{\alpha} = 0$ if $\alpha = 0, 1$, or $(\beta_1 - \beta_2)s_p^1 + \beta_2 - \alpha = 0$. Thus, $\dot{s}_p^1 = 0$ holds on the curve

$$l_1 : \alpha = \left\{1 - \frac{(a_{11} - a_{12})s_p^1 + a_{12}}{(a_{21} - a_{22})s_p^1 + a_{22}}\right\} s_p^1, \tag{12}$$

and $\dot{\alpha} = 0$ holds on the line

$$l_2 : \alpha = (\beta_1 - \beta_2)s_p^1 + \beta_2. \tag{13}$$

$\dot{s}_p^1 > 0$ (*resp*. < 0) holds above (*resp*. below) the curve l_1. $\dot{\alpha} < 0$ (*resp*. > 0) holds above (*resp*. below) the line l_2. The curve l_1 depends only on players' payoff matrix A while the line l_2 depends on the government's payoff matrices B and C. Figure 2 shows typical patterns of equilibrium points and schematics of vector fields.

The coordinates of the points V, T_1, T_0, T_{l_1}, and T_{l_2} are $(0,\beta_2)$, $(1,1)$, $(1,0)$, $\left(1, \frac{a_{21}-a_{11}}{a_{21}}\right)$, and $(1,\beta_1)$, respectively. Moreover, the s_p^1-coordinates of the point W, W_u, and W_s are solutions of

$$w_1\left(s_p^1\right)^2 + w_2 s_p^1 + a_{22}\beta_2 = 0, \tag{14}$$

where

$$w_1 = (a_{21} - a_{22})(\beta_1 - \beta_2) + (a_{11} - a_{12} - a_{21} + a_{22}), \tag{15}$$

$$w_2 = a_{22}\beta_1 + (a_{21} - 2a_{22})\beta_2 + (a_{12} - a_{22}). \tag{16}$$

Their α-coordinates are given by Eq. (13).

Figure 2(a) shows the case $\beta_1 > 1$. Under the condition $\frac{a_{21}-a_{11}}{a_{21}} < \beta_1 < 1$, Figs. 2(b), (c), and (d) show the cases $(w_2)^2 < 4a_{22}\beta_2 w_1$, $(w_2)^2 = 4a_{22}\beta_2 w_1$, and $(w_2)^2 > 4a_{22}\beta_2 w_1$, respectively. If the condition $\beta_1 < \frac{a_{21}-a_{11}}{a_{21}}$ holds, then we have Fig. 2(e).

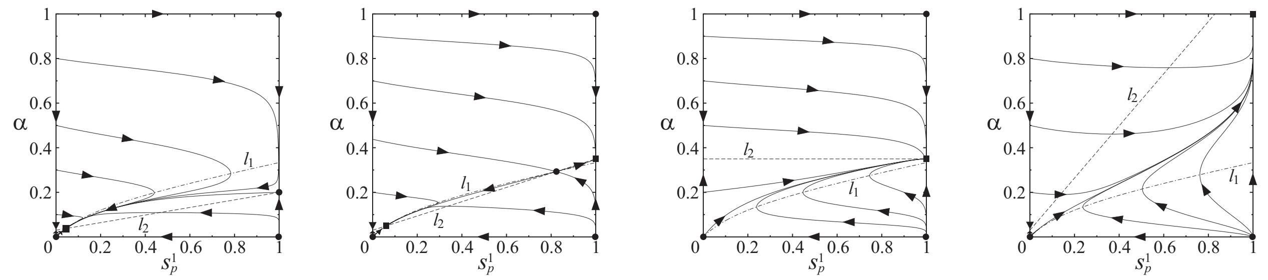
As shown in Fig. 2, stability conditions of points T_1, T_{l_2}, W, and W_s are given as follows:

- T_1 is asymptotically stable if $\beta_1 > 1$ (Fig. 2(a));
- T_{l_2} is asymptotically stable if $\frac{a_{21}-a_{11}}{a_{21}} < \beta_1 < 1$ (Figs. 2(b)–(d));
- W is stable if $\beta_1 < \frac{a_{21}-a_{11}}{a_{21}}$ (Fig. 2(e)); and
- W_s is stable if $\frac{a_{21}-a_{11}}{a_{21}} < \beta_1 < 1$ and $(w_2)^2 > 4a_{22}\beta_2 w_1$ (Fig. 2(d)).

The target state $s_p^* = (1, 0)^T$ corresponds to the boundary edge $\mathrm{T}_1\mathrm{T}_0$ in Fig. 2. Therefore, the achievement of the target state requires that the point T_{l_2} or T_1 is asymptotically stable.

If the stability condition of the point T_1 holds, then the government keeps increasing the intervention rate α to the complete intervention strategy $\alpha = 1$ since there is no reason for the government to reduce α. Although the target state is achieved, the original game structure is completely lost.

On the other hand, if the stability condition of T_{l_2} holds, then the target state is achieved and the intervention rate α keeps the intermediate level. Moreover, if the condition $(w_2)^2 < 4a_{22}\beta_2 w_1$ also holds, then T_{l_2} is globally asymptotically stable (Fig. 2(b)). To achieve the target state inde-



(i) $b_1 = 0.2$ and $b_2 = 0.03$ (ii) $b_1 = 0.35$ and $b_2 = 0.03$ (iii) $b_1 = 0.35$ and $b_2 = 0.35$ (iv) $b_1 = 1.2$ and $b_2 = 0.03$

Figure 3: Examples of phase portraits where the payoff matrices are defined by Eqs. (17) and (18).

pendent of initial population states of P, the cost-efficiency of the government's intervention not only to strategy 1 but also to strategy 2 must be sufficiently large.

If the cost-efficiency of the government's intervention to strategy 1 is less than $\frac{a_{21}-a_{11}}{a_{21}}$, then the cost of the intervention is so large that the government always wants to decrease the intervention rate. Thus the target state can not be achieved.

5. Example

Suppose that the payoff matrix A is given by

$$A = \begin{bmatrix} 4 & 0 \\ 6 & 1 \end{bmatrix}. \tag{17}$$

In such a game, players of P earn the maximum average payoff at the boundary point $s_p = (1, 0)^T$ and the minimum average payoff at the other boundary point $(0, 1)^T$. Figure 3 shows examples of its phase portraits, where

$$C = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{18}$$

The coordinates of the points T_{l_2}, V, and T_{l_1} are $(1, b_1)$, $(0, b_2)$, and $(1, \frac{1}{3})$, respectively. Note that $\beta_1 = b_1$ and $\beta_2 = b_2$ hold in this case. Black squares correspond to stable equilibrium points.

In the case that $b_1 = 0.2$ and $b_2 = 0.03$, since the point T_{l_2} is below the point T_{l_1}, the equilibrium points T_0, T_{l_2}, and T_1 which achieve the target state $s_p^* = (0, 1)^T$ are unstable, and the point W is a uniquely stable equilibrium point (see Figs. 3(i) and 2(e)).

In the case that $b_1 = 0.35$ and $b_2 = 0.03$, since the point T_{l_2} is above the point T_{l_1}, T_{l_2} is an asymptotically stable equilibrium point. However, the point W is also asymptotically stable since b_2 is not sufficiently large (see Figs. 3(ii) and 2(d)).

In the case that $b_1 = 0.35$ and $b_2 = 0.35$, since the point T_{l_2} is above the point T_{l_1} and the point V is sufficiently high, there exists no intersection of l_1 with l_2. Therefore, the point T_{l_2} is a globally asymptotically stable equilibrium point (see Figs. 3(iii) and 2(b)).

In the case that $b_1 = 1.2$ and $b_2 = 0.03$, the point T_{l_2} is above T_1, which is a globally asymptotically stable equilibrium point (see Figs. 3(iv) and 2(a)).

6. Conclusions

In this paper, we have defined the government's payoff as a sum of benefits which depend on the current state of the population and a cost of the government's intervention. Moreover, we have proposed a model which describes the evolution of its payoff reallocation strategy, and have investigated stability of its equilibrium points in the case that players have two strategies and the target state is on the boundary.

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