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Amplitude death in delayed chaotic systems coupled by diffusive connections

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Abstract—This paper studies amplitude death in delayed chaotic systems which are described by single scalar delay-differential-equations. They are coupled by three types of diffusive connections, such as normal, delayed, dynamic connections. A simple sufficient condition for avoiding death is derived. This condition is valid for delayed and dynamic connections. Numerical examples verify the sufficient condition.

1. Introduction

There have been various investigations of coupled nonlinear systems from the viewpoints of academic interests and practical applications [1]. Amplitude death in coupled nonlinear systems was considered as an attractive phenomenon [2, 3]. This phenomenon is a coupling-induced stabilization of a fixed point in the coupled *nonidentical* oscillators. Furthermore, it is guaranteed that death never occurs in coupled *identical* systems [3].

Reddy, Sen, and Johnston showed that death in coupled *identical* systems can be induced by a time-delay connection [4]. The delay-connection induced death has attracted a growing interest in the field of nonlinear dynamics [5]. It was observed experimentally in electronic circuits [6] and thermo-optical oscillators [7]. The theoretical analysis has been achieved by several researchers: stability of death in coupled simple oscillators near Hopf bifurcations [8], a sufficient condition (called the *odd number property*) under which death never occurs [9, 10], oscillators coupled by a one-way connection [11], death induced by dynamic connections [12], distributed delay effect [13], total and partial death in networks [14], coupled chaotic oscillators [15], and a ring of coupled limit cycle oscillators [16, 17].

Delayed chaotic systems described by single scalar delay-differential-equations have been widely used to study the infinite dimensional nonlinear phenomena [18]. In recent years, the coupled delayed-chaotic systems have created considerable interest from the viewpoints of synchronization [19], communication schemes [20], anticipation [21].

There have been few efforts to study the amplitude death in coupled high-dimensional chaotic systems. The present paper focuses on the delayed chaotic systems (i.e., hyper chaotic systems) coupled by three types of diffusive connections, such as normal connection, delayed connection [4, 10], and dynamic connection [12]. In electrical oscillators, these connections corresponds to resistance coupling, delay-line coupling, and resistance-capacitance coupling [12]. The previous study [10] provided a simple sufficient condition under which death in identical m-dimensional systems coupled by the delay connection never occurs for any coupling strength and delay time. The present paper also derives a simple sufficient condition for avoiding death in delayed chaotic systems coupled by three type connections.

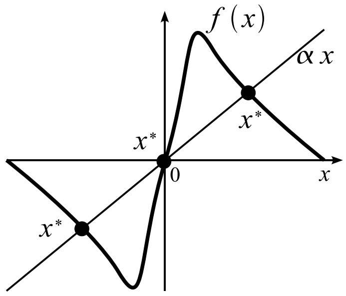


Figure 1: Nonlinear function and fixed points.

2. Delayed chaotic systems

Let us consider two identical delayed-chaotic systems

$$\begin{cases} \dot{x}_1 &= -\alpha x_1 + f(x_{1-\tau}) + u_1 \\ \dot{x}_2 &= -\alpha x_2 + f(x_{2-\tau}) + u_2 \end{cases}, \tag{1}$$

where $x_{1,2} \in \boldsymbol{R}$ are the system states, $f : \boldsymbol{R} \to \boldsymbol{R}$ is the nonlinear function, $x_{1,2-\tau} := x_{1,2}(t-\tau)$ are the delayed states, $u_{1,2} \in \boldsymbol{R}$ are the coupling signals. $\alpha > 0$ is the parameter. The fixed point of individual chaotic system without coupling (i.e., $u_{1,2} \equiv 0$) is given by

$$x^* : 0 = -\alpha x^* + f(x^*). \tag{2}$$

Figure 1 illustrates the nonlinear function f and the line αx. The fixed point x^* is located at the intersections of f and αx. It is assumed that the individual chaotic system without coupling behaves oscillatory or chaotically (i.e., x^* is unstable) throughout this paper.

This paper focuses on the three diffusive connections: normal connection, delayed connection [4, 10], and dynamic connection [12]. Figure 2 sketches the delayed

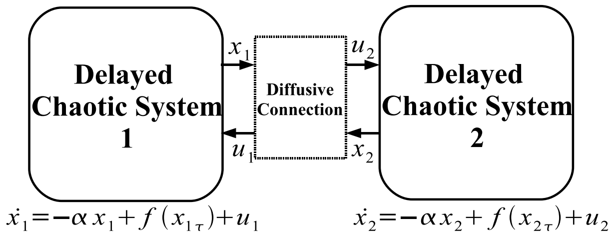


Figure 2: Illustration of delayed chaotic systems coupled by diffusive connections.

chaotic systems coupled by the diffusive connections. These connections illustrated in Fig. 3 are described as follows.

Normal connection (Fig. 3(a)):

$$u_{1,2} = k(x_{1,2} - x_{2,1}). \tag{3}$$

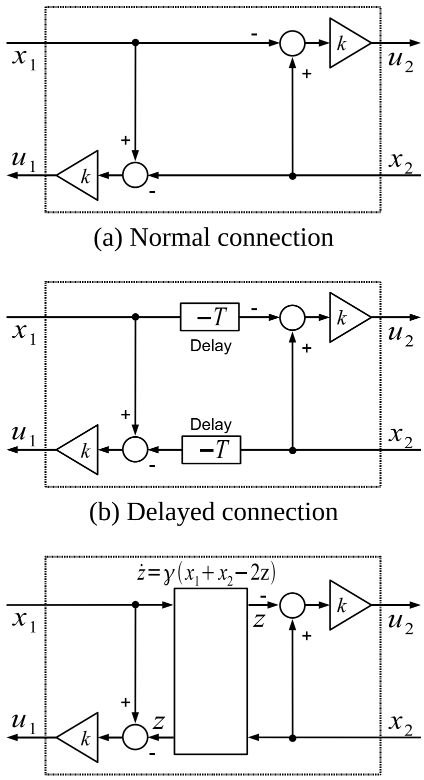
Delayed connection (Fig. 3(b)):

$$u_{1,2} = k(x_{1,2} - x_{2,1-T}). \tag{4}$$

Dynamic connection (Fig. 3(c)):

$$\begin{cases} u_{1,2} &= k(x_{1,2} - z) \\ \dot{z} &= \gamma(x_1 + x_2 - 2z) \end{cases}. \tag{5}$$

The delayed states are denoted by $x_{2,1-T} := x_{2,1}(t-T)$, and $\gamma > 0$ is the parameter. $T \geq 0$ is the delay time for coupling. It should be noted that these connections change the stability of the fixed point x^*. However, they never move its location. The amplitude death can be explained as a diffusive-connection induced stabilization of the unstable fixed point x^*.



(a) Normal connection

(b) Delayed connection

(c) Dynamic connection

Figure 3: Block diagrams of three-type diffusive connections.

3. Stability analysis

The stability of the fixed point x^* in the delayed chaotic systems coupled by normal (3), delayed (4), and dynamic (5) connections is investigated. The linear stability analysis allows us to obtain the characteristic equation $g^{(1)}(\lambda)g^{(2)}(\lambda) = 0$, where $g^{(1)}(\lambda)$ and $g^{(2)}(\lambda)$ depend on the connection type as follows.

Normal connection:

$$\begin{aligned} g^{(1)}(\lambda) &= \lambda + \alpha - \beta(x^*)e^{-\lambda\tau}, \\ g^{(2)}(\lambda) &= \lambda + \alpha - 2k - \beta(x^*)e^{-\lambda\tau}. \end{aligned}$$

Delayed connection:

$$\begin{aligned} g^{(1)}(\lambda) &= \lambda + \alpha - k(1 - e^{-\lambda T}) - \beta(x^*)e^{-\lambda\tau}, \\ g^{(2)}(\lambda) &= \lambda + \alpha - k(1 + e^{-\lambda T}) - \beta(x^*)e^{-\lambda\tau}. \end{aligned}$$

Dynamic connection:

$$\begin{aligned} g^{(1)}(\lambda) &= \lambda + \alpha - k - \beta(x^*)e^{-\lambda\tau}, \\ g^{(2)}(\lambda) &= \det \begin{bmatrix} \lambda + \alpha - k - \beta(x^*)e^{-\lambda\tau} & 2k \\ -\gamma & \lambda + 2\gamma \end{bmatrix}. \end{aligned}$$

$\beta(x^*) := \{\mathrm{d}f(x)/\mathrm{d}x\}_{x=x^*}$ is the slope of $f(x)$ at x^*. From these characteristic equations, a simple instability condition can be easily derived.

Lemma 1. *The amplitude death at the fixed point x^* never occurs in delayed chaotic systems coupled by normal connection (3) for any $k \in \boldsymbol{R}$.*

Proof. It is noticed that $g^{(1)}(\lambda) = 0$ is identical with the characteristic equation of the individual chaotic system without coupling. According to our assumption (i.e., x^* without coupling is unstable), $g^{(1)}(\lambda) = 0$ has at least one root in open right half complex plane. Since $g^{(1)}(\lambda)$ does not depend on k, death never occurs for any $k \in \boldsymbol{R}$. □

This lemma can be considered as an extension of Lemma 1 in [10] to delayed chaotic systems. These results indicate that the normal diffusive connection never induces the amplitude death whether individual system includes delayed state or not.

According to the previous results in [10], we know the fact that if $\lim_{\lambda\to+\infty} h(\lambda) = +\infty$ and $h(0) < 0$, then there exits at least one positive real root for $h(\lambda) = 0$. This fact leads us to obtain the following two theorems.

Theorem 1. *If $\alpha < \beta(x^*)$, then the amplitude death at the fixed point x^* never occurs in delayed chaotic systems coupled by* **delayed** *connection (4) for any $k \in \boldsymbol{R}, T > 0$, and $\tau \geq 0$.*

Proof. It is obvious that $\lim_{\lambda\to+\infty} g^{(1)}(\lambda) = +\infty$. $g^{(1)}(0) = \alpha - \beta(x^*)$ does not depend on k, T, τ. If $g^{(1)}(0) < 0$, then $g^{(1)}(\lambda) = 0$ has at least one positive real root. Therefore, the amplitude death never occurs for any k, T, τ when $\alpha < \beta(x^*)$ holds. □

Theorem 1 is an extension of Theorem 2 in [10] to delayed chaotic systems; hence, $\alpha < \beta(x^*)$ corresponds to the odd number property in [10].

Theorem 2. *If $\alpha < \beta(x^*)$, then the amplitude death at the fixed point x^* never occurs in delayed chaotic systems coupled by* **dynamic** *connection (5) for any $k \in \boldsymbol{R}, \gamma > 0$, and $\tau \geq 0$.*

Proof. It is clear that $\lim_{\lambda\to+\infty} g^{(1)}(\lambda) = +\infty$. The parameters k, γ, τ are independent of $g^{(1)}(0) = \alpha - \beta(x^*)$. Then $g^{(1)}(\lambda) = 0$ has at least one positive real root for any $k \in \boldsymbol{R}, \gamma > 0$, $\tau \geq 0$ when $g^{(1)}(0) < 0$. Therefore, the amplitude death never occurs if $\alpha < \beta(x^*)$. □

This theorem is also an extension of the previous result in [12]. Remark that these theorems provide just sufficient condition for avoiding death. In other words, when $\alpha > \beta(x^*)$ is held, the theorems cannot guarantee whether death occurs or not.

Now we consider the case where the number of delayed chaotic systems is one. In this case, death induced by delayed connection (4) and dynamic connection (5) can be regarded as a stabilization of delayed chaotic systems by delayed feedback control and dynamic feedback control [22] respectively. From Theorems 1 and 2, we can easily derive the following result.

Corollary 1. *Consider a single delayed-chaotic system:*

$$\dot{x} = -\alpha x + f(x_\tau) + u.$$

If $\alpha < \beta(x^*)$, then the fixed point x^* is not stabilized by delayed feedback control,

$$u = k(x - x_T),$$

for any $k \in \boldsymbol{R}$, $T > 0$, $\tau \geq 0$. In addition, if $\alpha < \beta(x^*)$, then x^* is not stabilized by dynamic feedback control,

$$u = k(x - z),\ \ \dot{z} = \gamma(x - z),$$

for any $k \in \boldsymbol{R}$, $\gamma > 0$, $\tau \geq 0$.

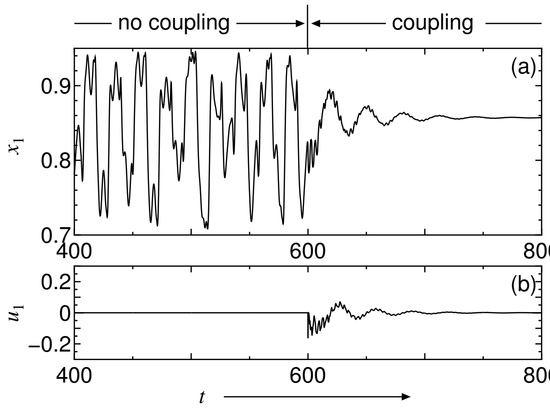


Figure 4: Amplitude death in the delayed chaotic systems coupled by delayed connection ($k = -6$, $T = 1.55$).

4. Numerical examples

Let us employ the following nonlinear function:

$$f(x) = \begin{cases} 0 & x \leq -4/3, \\ -1.8x - 2.4 & -4/3 < x \leq -0.8, \\ 1.2x & -0.8 < x \leq 0.8, \\ -1.8x + 2.4 & 0.8 < x \leq 4/3, \\ 0 & x > 4/3. \end{cases}$$

It was reported that the hyperchaos occurs in Eq. (1) without coupling at $\alpha = 1.0$ and $\tau = 10.0$ [23]. The function f has the three fixed points:

$$x^* = -6/7,\ \ 0,\ \ +6/7.$$

For the fixed points $x^* = \pm 6/7$, the slope of f at x^* is $\beta(x^*) = -1.8$. As Theorems 1 and 2 are not satisfied, it cannot be guaranteed whether death occurs or not. On the other hands, the slope at $x^* = 0$ is $\beta(x^*) = 1.2$. Since Theorems 1 and 2 are satisfied, we notice that delayed (4) and dynamic (5) connections never induce death at $x^* = 0$ for any k, T, τ, γ.

Figure 4 shows the state x_1 and the coupling signal u_1 of Eq. (1) with delayed connection (4). It can be seen that death occurs at $x^* = +6/7$. Furthermore, as shown in Fig. 5, x_1 and u_1 in Eq. (1) with dynamic connection (5) converge on $x^* = +6/7$ and $u_1 = 0$ respectively. From these observations, we see that delayed (4) and dynamic (5) connections induce death at $x^* = +6/7$, but not at $x^* = 0$. These numerical results do not contradict Theorems 1 and 2.

5. Conclusions

This paper investigates the amplitude death in the delayed chaotic systems coupled by three types of diffusive connections. A simple sufficient condition for avoiding death is derived. It is shown that the condition is valid for the numerical examples.

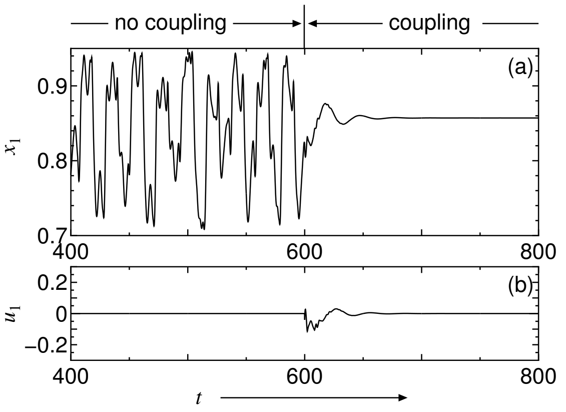


Figure 5: Amplitude death in the delayed chaotic systems coupled by dynamic connection ($k = -5$, $\gamma = 0.2$).

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