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An Investigation on the Absolute Stability of Discrete and Continuous Time Lur'e Systems

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Abstract—In this paper sufficent conditions for the absolute stability of discrete time and continuous time single variable Lur'e systems are investigated numerically. These conditions are expressed as simple restrictions on the root locus of the system. If these conditions hold it is shown that a Liapunov function exists for the system.

1. Introduction

In recent years there has been rapid development in the study of switched systems, giving rise to many questions regarding their stability. A switched control system is a type of hybrid dynamical system [1] consisting of continuous and/or discrete time processes interfaced with some decision making process, which estimates the current active environment and selects the appropriate controller. Switched systems have many practical applications in both electrical and mechanical control systems. Applications include power systems, power electronics, air traffic control, aircraft and satellites [2],[3]. The technique of switching among different controllers or switching into different states can provide a significant improvement in performance compared to that of a fixed controller [4].

The type of dynamical systems investigated here are:
1) Discrete time 2nd order single variable Lur'e systems of the form

$$x_{i+1} = \tilde{A}x_i - k_i\tilde{b}\tilde{c}^T x_i, \quad |k_i| \leq 1 \tag{1}$$

where, $\tilde{A} \in \Re^{2\times 2}$, $\tilde{b}, \tilde{c} \in \Re^2$ are linearly independent constant vectors of the form $\tilde{c}^T = \begin{bmatrix} c_2 & c_1 \end{bmatrix} \in \Re^2$ and $\tilde{b}^T = \begin{bmatrix} 0 & 1 \end{bmatrix} \in \Re^2$ and

$$\tilde{G}(z) = \tilde{c}^T(zI - A)^{-1}\tilde{b} = \frac{\tilde{N}(z)}{\tilde{D}(z)}.$$

2) Continuous time nth order single variable Lur'e systems of the form

$$\dot{x} = Ax - k(x,t)bc^T x, \quad k(x,t) \geq 0 \tag{2}$$

where, $A \in \Re^{n\times n}$, $b, c \in \Re^n$ are linearly independent constant vectors of the form $c^T = \begin{bmatrix} \hat{c}^T & 1 \end{bmatrix}$, $\hat{c} \in \Re^{n-1}$ and $b^T = \begin{bmatrix} \hat{0}^T & 1 \end{bmatrix}$, $\hat{0} \in \Re^{n-1}$ and

$$G(s) = c^T(sI - A)^{-1}b = \frac{N(s)}{D(s)}.$$

The problem of interest is the absolute stability of class (1) and (2). Many stability criteria have been derived for Lur'e systems. One criterion of particular importance is the well known Circle Criterion [5],[6],[7] since it applies to time-varying systems and is equivalent to the existence of a common quadratic Liapunov function [8]. In recent years, much research on the stability of Lur'e systems has focused on constructing common Liapunov functions. Molchanov and Pyatnitski [9] establish a necessary and sufficient condition for absolute stability of time-varying Lur'e systems is the existence a common Liapunov function. However testing the existence of these common Liapunov functions requires the use of numerical procedures in general [10].

In [11],[12],[13] 2nd order Lur'e systems of the form (2) are investigated. Wulff *et al.*[12] establish absolute stability conditions that have a simple geometrical form and are therefore readily tested. If the roots of the polynomial $D(s) + \sigma N(s)$ for all $\sigma \geq 0$ are restricted to the 45^0 region (see Figure 1), a common unic Liapunov function or a common quadratic Liapunov function exists. In [13] it is shown that if the roots of the polynomial $D(s) + \sigma N(s)$ for all $\sigma \geq 0$ lie in the 45^0 region a common piecewise quadratic Liapunov function exists.

In this paper absolute stability conjectures for class (1) and (2) are investigated numerically. These conjectures all have a simple geometrical form. For class (1) numerical investigations suggest that if the roots of $\tilde{D}(z) + \sigma\tilde{N}(z)$ lie in the Discrete-45^0 region, i.e. the discrete time analog of the 45^0 region (see Figure 2), for $|\sigma| \leq 1$ then class (1) is absolutely stable and a

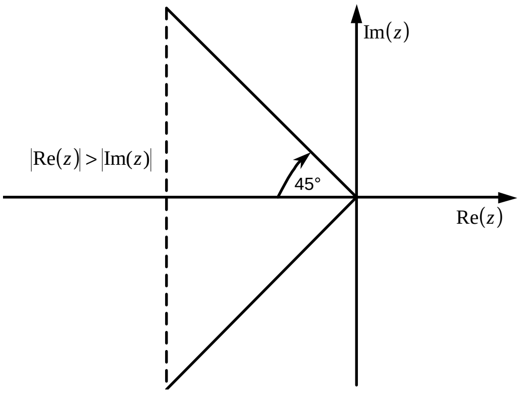

Figure 1: The 45^0 region

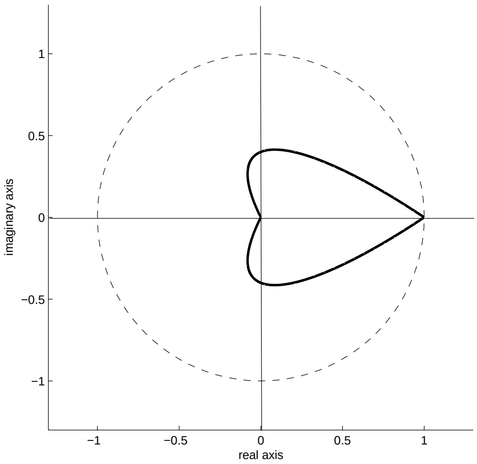

Figure 2: The Discrete-45^0 region

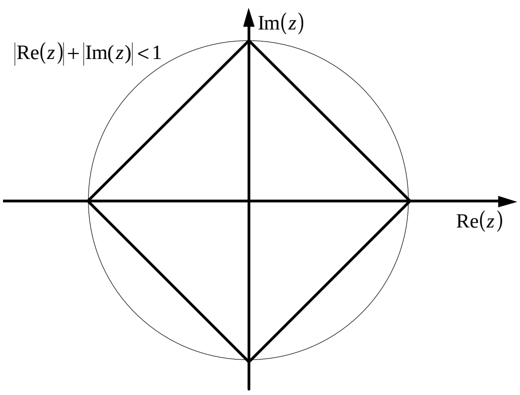

Figure 3: The Diamond region

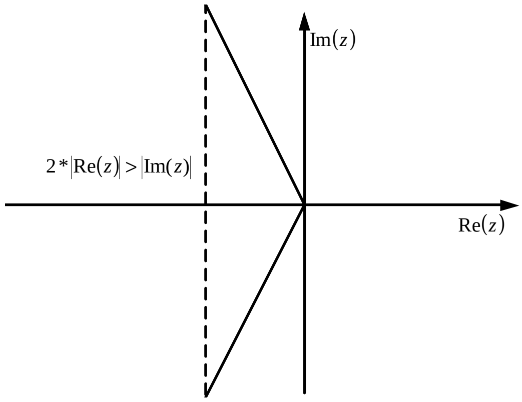

Figure 4: The Super-45^0 region

unic Liapunov function exists. For class (1) another stability region, the Diamond region (see Figure 3) is investigated. Numerical investigations suggest that if the roots of $\tilde{D}(z) + \sigma\tilde{N}(z)$ lie in the Diamond region for $|\sigma| \leq 1$ then class (1) is absolutely stable and a common piecewise quadratic Liapunov function exists. For class (2) with $n = 2$ and $n = 3$ numerical investigations show that if roots of $D(s) + \sigma N(s)$ lie in the Super-45^0 region for $\sigma \geq 0$, (see Figure 4), then class (2) is absolutely stable.

2. Mathematical Preliminaries

In this section some useful definitions are presented.

Definition 1: Discrete-45^0 Condition
The roots of polynomial $\tilde{D}(z) + \sigma\tilde{N}(z)$ lie in the Discrete-45^0 region for all $|\sigma| \leq 1$.

Definition 2: Diamond Condition
The roots of polynomial $\tilde{D}(z) + \sigma\tilde{N}(z)$ lie in the Diamond region for all $|\sigma| \leq 1$.

Definition 3: Super-45^0 Condition
The roots of polynomial $D(s) + \sigma N(s)$ lie in the Super-45^0 region for all $\sigma \geq 0$.

3. The Discrete-45^0 Condition and the Absolute Stability of 2nd order discrete time Lur'e systems

In this section examples of discrete time Lur'e systems of the form (1) are investigated numerically. The aim is to show using numerical investigations that the Discrete-45^0 condition is a sufficient condition for absolute stability. This will also provide strong evidence as to a possible analytical proof of this condition.

The absolute stability of these systems are investigated using the following method.

Method 1:
Polanski [14] has formulated the search for unic Liapunov functions as a linear programming problem. Using this method we determine whether a Liapunov function of the form

$$V(x_i) = \|Mx_i\|_1 \tag{3}$$

exists for class (1) where $M \in \Re^{2\times 2}$. (3) is a Liapunov function for class (1) if there exists a matrix Q such that

$$M\tilde{A} - QM = 0 \quad \text{and} \quad \|Q\|_1 < 0. \tag{4}$$

Results:
Five thousand Lur'e systems of the form (1) that satisfy the Discrete-45^0 condition were randomly generated. For half of these systems the eigenvalues of $\tilde{A}$ are real and for half the eigenvalues of $\tilde{A}$ are a complex conjugate pair. By Method 1 99 percent of systems tested were found to be absolutely stable. The remaining 1 percent of systems were found to be absolutely stable by the discrete-time Circle Criterion.

These results provide strong evidence that for class (1) the Discrete-45^0 condition implies absolute stability. They also suggest that the Discrete-45^0 condition implies the existence of a common unic Liapunov function or a common quadratic Liapunov function.

4. The Diamond Condition and the Absolute Stability of 2nd order discrete time Lur'e systems

In this section examples of discrete time Lur'e systems of the form (1) are once again investigated numerically. The motivation being to investigate whether the Diamond condition is a sufficient condition for absolute stability and to gain insight into how to prove this condition.

The absolute stability of these systems are investigated using the following method.

Method 2:
In [15],[16] the search for common piecewise quadratic Liapunov functions is formulated as a convex optimization problem which is expressed in terms of linear matrix inequalities (LMI's). Using this method we determine whether a common piecewise quadratic Liapunov function of the form

$$V(x_i) = \begin{cases} {x_i}^T P_1 x_i & \text{for } (b^T x_i)(c^T x_i) > 0 \\ {x_i}^T P_2 x_i & \text{for } (b^T x_i)(c^T x_i) < 0. \end{cases} \tag{5}$$

where $P_1 = P_1^T \in \Re^{2\times 2}$ and $P_2 = P_2^T \in \Re^{2\times 2}$ exists for class (1).

Results:
Five thousand Lur'e systems of the form (1) that satisfy the Diamond condition were randomly generated. For half of these systems the eigenvalues of $\tilde{A}$ are real and for half the eigenvalues of $\tilde{A}$ are a complex conjugate pair. By Method 2 all systems were found to be absolutely stable.

These results provide strong evidence that for class (1) the Diamond condition implies absolute stability. They also suggest that the Diamond condition implies the existence of a common piecewise quadratic Liapunov function.

5. The Super-45^0 Condition and the Absolute Stability of continuous time Lur'e systems

In this section examples of 2nd order and 3rd order continuous time Lur'e systems of the form (2) are investigated numerically. Since previous investigations have shown the importance of the 45^0 region in establishing the absolute stability of class (2), we investigated whether the absolute stability of this class could be established using a larger region, the Super-45^0 region.

The absolute stability of these systems are investigated using the following method.

Method 3:
Using the method of Johansson and Rantzer [15] we determine whether a common piecewise quadratic Liapunov function of the form

$$V(x) = \begin{cases} x^T P_1 x & \text{for } (b^T x)(c^T x) > 0 \\ x^T P_2 x & \text{for } (b^T x)(c^T x) < 0. \end{cases} \tag{6}$$

where $P_1 = P_1^T \in \Re^{2\times 2}$ and $P_2 = P_2^T \in \Re^{2\times 2}$ exists for class (2).

Results:
Five thousand 2nd order Lur'e systems of the form (2) that satisfy the Super-45^0 condition were randomly generated. For half of these systems the eigenvalues of A are real and for half the eigenvalues of A are a complex conjugate pair. By Method 3 all systems were found to be absolutely stable.

Five thousand 3rd Lur'e systems of the form (2) that satisfy the Super-45^0 condition were randomly generated. For half of these systems the eigenvalues of A are real and for half the eigenvalues of A contain a complex conjugate pair. By Method 3 all systems were found to be absolutely stable.

These results provide strong evidence that for class (2) with $n = 2$ and $n = 3$ the Super-45^0 condition implies absolute stability. They also suggest that the Super-45^0 condition implies the existence of a common piecewise quadratic Liapunov function.

6. Conclusions and Future Work

Sufficient conditions for the absolute stability of low order discrete time and continuous time Lur'e system are investigated numerically. These conditions have a simple geometrical form. For thousands of randomly generated systems of the form (1) the Discrete-45^0 condition implies absolute stability by the existence of a unic Liapunov function. Similarly the Diamond Condition implies absolute stability of randomly generated systems of the form (1) by the existence of a common piecewise quadratic Liapunov function. For 2nd and 3rd order continuous time systems of the form (2) numerical investigations show that the Super-45^0 condition implies absolute stability by the existence of a common piecewise quadratic Liapunov function.

Given this numerical evidence future work will focus on proving that the Discrete-45^0 condition and the Diamond Condition are indeed sufficient conditions for the absolute stability of class (1) and that the Super-45^0 condition is a sufficient condition for the absolute stability of class (2).

Acknowledgments

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