A Two-Level Spectral Preconditioning for the Finite Element Method

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***Abstract*-An efficient auxiliary space preconditioning (ASP) was proposed for the linear matrix equation that was formed by the frequency-domain finite element method. A new two-level spectral preconditioning utilizing auxiliary space preconditioning is presented to solve the linear system. This technique is a combination of ASP and a low-rank update spectral preconditioning, in which the restarted deflated generalized minimal residual GMRES with the newly constructed spectral two-step preconditioning is considered as the iterative method for solving the system. Numerical experiments indicate that the proposed preconditioning is efficient and can significantly reduce the iteration number.**

I. INTRODUCTION

The finite element method (FEM) has been applied to the analysis of problems in electromagnetics for more than 35 years. A large number of research papers can be found in the literature [1]-[4]. The application of the finite-element method to electromagnetic problems often yields a sparse, symmetric, and very high-order system of linear algebraic equations. These highly sparse large linear equations can be solved using efficient solution techniques for sparse matrices based on iterative methods.

The Krylov subspace iterative methods like the generalized minimal residual (GMRES) method converges much faster than the other methods. The number of iteration required in the GMRES method can be controlled to some degree by the use of various preconditioning strategies. It is then desirable to precondition the coefficient matrix so that the modified system is well conditioned and can converge to an exact solution in significantly fewer iteration numbers than the original system. Many scholars have done a lot of researches on improving the efficiency of the iterative solution in the past few decades. The incomplete factorizations of the coefficient matrix and its block variants are a widely used class of preconditioners [5]-[7]. The multigrid preconditioning has also widely been used for the FEM. The geometric multigrid (GMG) method is the earliest form of multigrid, originally invented for finite difference methods [8]. The algebraic multigrid (AMG) method [9] was developed to overcome this limitation of GMG. It operates more on the level of the matrix than of the underlying FE mesh, which need not be nested.

In most of the cases, a single preconditioning can improve the iteration convergence speed to a certain extent. We can get more obvious convergence improvements when combining different preconditionings. In this paper, we apply the auxiliary space preconditioning (ASP), which was proposed in [10], as the first-step preconditioning. A spectral preconditioning [11]-[13] was applied in a two-step manner that attempts to further enhance the quality of the one-step preconditioning, resulting in a faster convergence rate.

This paper is organized as follows. Section Ⅱ gives an introduction to the two-level spectral preconditioning utilizing ASP in detail. Numerical experiments are presented to show the efficiency of the spectral two-step preconditioning in SectionⅢ. SectionⅣ gives some conclusions and comments.

II. HHEORY

A. The Auxiliary Space Preconditioning

We consider solving a large sparse linear system

$$Ax = b \tag{1}$$

With $A \in C^{n \times n}$, b, $x \in C^n$, which arises from finite-element discretization of the Helmholtz boundary value problems, where A is complex and symmetric and x is a column vector of the unknown values of E on the element edges.

The auxiliary space preconditioner is an approximate inverse of A , used to improve the convergence of an iterative solution to Ax=b using a Krylov method. Let V be the space spanned by the lowest order edge element basis functions on a tetrahedral mesh. There is an associated space N of piecewise linear, scalar functions on the same mesh and it is well known that ∇N is a subspace of [15]. We also need the space of vector "nodal" functions, N^3 . These are vector functions that, unlike edge basis functions, are both normally and tangentially continuous from one element to the next and are fully first-order in each element.

These spaces are linked by the following result [16]: for any $E \in V$ there exists $u \in N^3$ and $\varphi \in N$ such that:

$$E = E_S + \Pi u + \nabla \varphi \tag{2}$$

Where E_S is a "small" component in V. This suggests that it might be possible to solve the problem (1) approximately by solving related problems on the auxiliary spaces N^3 and N.

The operator ∇ becomes the sparse matrix G , which is simply the "node-to-edge" mapping matrix with entries of -1 and 1 per row . To preserve symmetry, we use the transpose, G^T , to map backwards, from V to N.

For N^3 we use the N basis for each Cartesian component of the vector and represent $u \in N^3$ by a column vector in which nodal values of the x , y and z components occupy 3 successive blocks. Then the operator Π becomes a sparse matrix which also has a block form:

$$\Pi = [\Pi_x \quad \Pi_y \quad \Pi_z] \tag{3}$$

Each block has the same dimension and sparsity pattern as G, i.e., two nonzero entries per row. It can be shown that the two nonzero values for row are identical, and, for Π_x , are equal to $(\frac{1}{2} G x_C)_i$, where x_C is a vector containing the x coordinates of the nodes; similarly for Π_y and Π_z . We use Π^T to map backwards, from V to N^3 .

None of these approximations on its own is very effective, but the following combined approach might be better:

$$A^{-1} r \cong R_f(A) D R_b(A) r + \sum_{i=x,y,z} \Pi_i B_i \Pi_i^T r + G B_n G^T r \tag{4}$$

Where $R_f(A) \triangleq (D+L)^{-1}, R_b(A) = (D+U)^{-1}$, $A_n \triangleq G^T A G$, $A_x \triangleq \Pi_x^T A \Pi_x$, $A_y \triangleq \Pi_y^T A \Pi_y$, $A_z \triangleq \Pi_z^T A \Pi_z$, We call the approximate inverses of these four matrices B_n , B_x , B_y and B_z , respectively. D is the diagonal part of A, L and U are the strict lower and upper triangular parts of A.

A W-cycle has been used. Omitting the "Residual update" lines for brevity, this is:

Backward GS: $\Delta x \leftarrow R_b(A) r$

Auxiliary spaces: $\Delta x \leftarrow \sum_{i=x,y,z} \Pi_i B_i \Pi_i^T r + G B_n G^T r$

Backward GS: $\Delta x \leftarrow R_b(A) r$

Auxiliary spaces: $\Delta x \leftarrow \sum_{i=x,y,z} \Pi_i B_i \Pi_i^T r + G B_n G^T r$

Forward GS: $\Delta x \leftarrow R_f(A) r$

Auxiliary spaces: $\Delta x \leftarrow \sum_{i=x,y,z} \Pi_i B_i \Pi_i^T r + G B_n G^T r$

Forward GS: $\Delta x \leftarrow R_f(A) r$

The approximate inverses of A defined by the W-cycle algorithms is the auxiliary space preconditioner considered in this paper.

B. *The Spectral Low Rank Preconditioning*

Although the ASP described above is very effective as shown in [10], the construction of it is inherently local. When the exact inverse of the original matrix is globally coupled, this lack of global information may have a severe impact on the quality of the preconditioner. We can get more obvious convergence improvements if recovering global information. In this case, some suitable mechanism has to be considered to recover global information.

We firstly let the most of eigenvalues of the system concentrate on the unit 1 by using the ASP, which eliminates the high frequency component of iteration process and accelerates the iteration convergence speed. A spectral preconditioner proposed in [14] can be introduced and used in a two-step way for the above ASP preconditioned system. The purpose here is to recover global information by removing the effect of some smallest eigenvalues in magnitude in the auxiliary space preconditioned matrix, which potentially can slow down the convergence of Krylov solvers.

Suppose $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of the ASP preconditioned matrix $M_1 A$ from small to large, U be a set of eigenvectors of dimension k associated with the smallest eigenvalues of the ASP preconditioned matrix $M_1 A$.

Define the second spectral preconditioner as:

$$M_2 = I_n + U(1/|\lambda_n| T - I_k) U^H \tag{5}$$

Where $T = U^H (M_1 Z) U$, I_n and I_k are unit matrix of dimension N and K respectively, and $\lambda_{k+1}, \lambda_{k+2}, ..., \lambda_n, |\lambda_n|, ...|\lambda_n|$ are the eigenvalues of the coefficient matrix $M_2 M_1 A$.

From the above analysis, we can convert the K smallest eigenvalues of the coefficient matrix $M_1 A$'s characteristic spectrum which is based on ASP preconditioner to K arithmetic numbers whose values are $|\lambda_n|$. This process can eliminate negative influences of the K smallest eigenvalues. Combining the second preconditioning with the previously preconditioning in a two-step manner, a new two-step preconditioning is derived and has the form of:

$$M_2 M_1 A x = M_2 M_1 b \tag{6}$$

Supposing that M_1 is a preconditioner of A, M_2 is a preconditioner of $M_1 A$. Therefore, a new two-level spectral preconditioning of multilevel fast multipole method is presented, which is a combination of an ASP and a spectral preconditioner, as follows:

(1) Firstly, construct the auxiliary space preconditioner M_1 using the matrix element of the matrix A, and then solve the K smallest eigenvalues of the linear equations (1) after the preconditioner M_1 by GMRES-DR iterative algorithm.

(2) Secondly, construct the spectrum preconditioner $M_2 = I_n + U(1/|\lambda_n| T - I_k) U^H$ using the information of eigenvectors.

(3) Solve the linear equations (6) by the two-step preconditioner iteration.

III. Numerical Results

Firstly, scattering by a dielectric sphere with radius of 60 mm is considered to show the correctness of the proposed method. There are 108331 unknowns after discretization. The incident plane wave direction is fixed at $\theta^{inc} = 0^\circ, \phi^{inc} = 0^\circ$, the frequency is 1GHz， and the scattering angle is fixed at $\theta_s = 0^\circ - 180^\circ, \phi_s = 90^\circ$. The dielectric constant is $\varepsilon_r = 2\varepsilon_0$. Locally-conformal PML is used in this test.

As shown in Fig.1, the comparison is made for the bistatic RCS of vertical polarization. It can be found that there is an excellent agreement between them and this demonstrates the validation of the proposed algorithm. The convergence history is given in Fig.2.

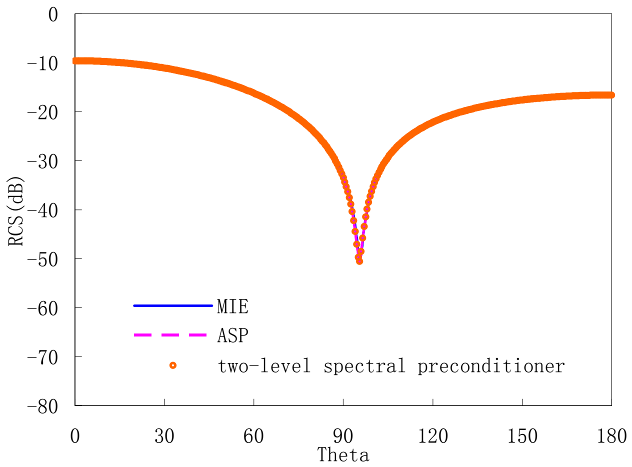

Figure 1. Bistatic RCS of the dielectric sphere

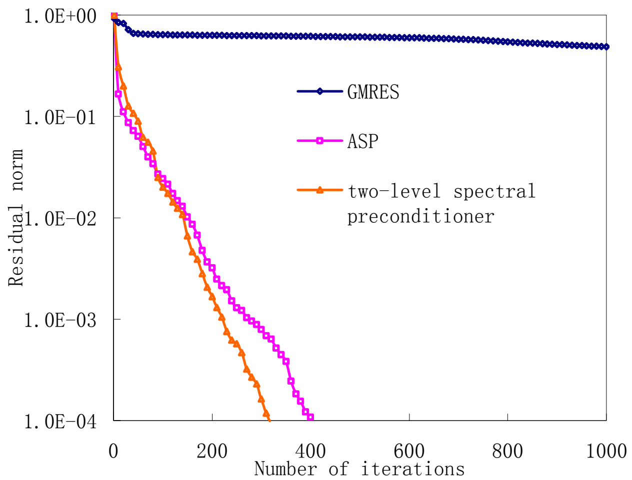

Figure 2. Convergence history of GMRES algorithms for the dielectric sphere

The second example is an analysis of a three-dimensional (3-D) discontinuity of a waveguide partially filled with a dielectric, which is shown in Fig.3.The rectangular waveguide has a width of a = 2 and height of b = 1; the inserted dielectric material slab has dimensions c = 0.888, d = 0.399, and w = 0.8; and the dielectric constant $\varepsilon_r = 6\varepsilon_0$. In this edge-FEM three-dimensional simulation, in order to obtain the reflection coefficient, one block of perfectly matched layer (PML) is placed at the waveguide output to simulate the matched output load. The use of PML in computational domains significantly deteriorates the condition number of the resulting FEM system. A total of 39817 unknown edges are to be solved in a large, sparse matrix equation. The convergence history at 9GHz is given in Fig.4. It can be found that when compared with the ASP preconditioned method, the two-step spectral preconditioned method decreases the number of iterations by a factor of 2.67. Larger improvements can also be found when compared with the GMRES method without preconditioning in terms of iterations.

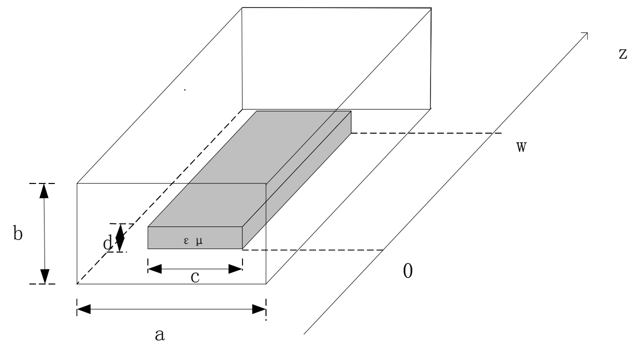

Figure3. Configuration of a dielectric partially filled rectangular waveguide

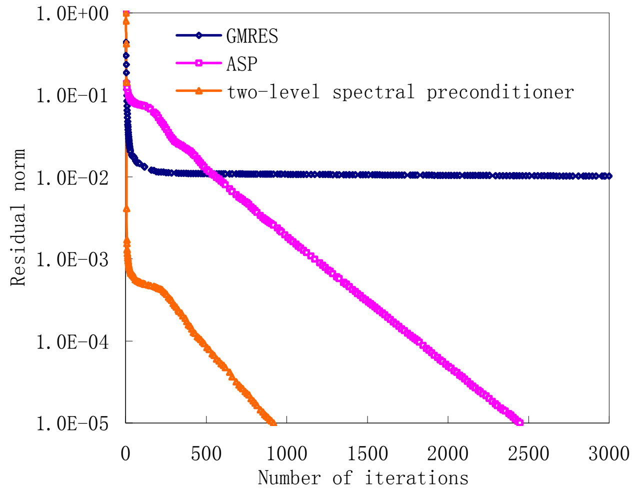

Figure 4. Convergence history of GMRES algorithms for the waveguide

IV. Conclusion

In this paper, a two-level spectral preconditioning utilizing ASP is proposed for FEM. The key of the paper is to combine the spectral preconditioner and the auxiliary space preconditioner in the two-step manner, resulting in faster convergence for GMRES iterations. The right-hand side system is solved by use of the GMRES-DR algorithm and the approximate smallest eigenvector information is obtained for constructing the spectral preconditioner for the system. Numerical experiments are performed and comparisons are made in the numerical results. It can be found that the proposed two-level spectral preconditioner utilizing ASP is more efficient and can significantly reduce the overall simulation time.

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