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A Dissipated Power-Based Analysis of Frequency Entrainment Described by van der Pol and PLL Equations

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Abstract—This paper performs an analysis of frequency entrainment described by van der Pol and PLL equations based on dissipated power. The entrainment originates from two different types of limit cycle. To explore the relationship between the geometry of limit cycle and the stability of entrainment, we investigate the entrainment by applying *dissipated power*. The power corresponds to dissipation part of averaged potential by Kuramitsu *et al*. We show response curves for dissipated power. The result implies that the magnitude of dissipated power is directly related to stability in the equations.

1. Introduction

Frequency entrainment has been extensively studied in nonlinear dynamical systems interacting self-sustained oscillators [1, 2, 3]. The oscillators, exchanging their stored energy through the interaction, synchronize. Therefore, energy exchange and balance in dynamical systems are crucial for understanding frequency entrainment. However, the energy aspect has not been fully used for analyzing frequency entrainment except a few references [4, 5, 6].

We have performed an energy-based analysis of frequency entrainment described by van der Pol and PLL equations [7]. One purpose of the analysis is to clarify the mechanism of entrainment with different types of limit cycle: libration [8] for van der Pol equation and rotation for PLL equation. Response curves for energy supplied by driving oscillatory forces are shown in [7] to explore the mechanism.

This paper provides an analysis of the frequency entrainment based on *dissipated power*. The dissipated power coincides with the averaged potential for dissipation elements in nonlinear circuits proposed by Kuramitsu *et al* [5, 6]. The averaged potential is applied to analysis of frequency entrainment. Response curves for dissipated power are investigated numerically and analytically. Finally, we show that the magnitude of dissipated power is related to stability in the equations.

2. Model Equations

In this section, van der Pol and PLL equations are reintroduced with the definitions of energy balance relation.

2.1. Van der Pol Equation

The van der Pol equation [2, 9] is well known as

$$\begin{cases} \dot{u} = v, \\ \dot{v} = \mu(1 - \beta u - \gamma u^2)v - u + B\cos\nu t, \end{cases} \tag{1}$$

where μ denotes the small positive parameter. β and γ depict the fixed parameters. $B\cos\nu t$ corresponds to the driving oscillatory force. The overdot denotes the differentiation with respect to t. Here, a smooth function $S_{\mathrm{v}}(u, v)$ is defined as

$$S_{\mathrm{v}}(u, v) = \frac{v^2}{2} + \frac{u^2}{2}. \tag{2}$$

S_{v} is called storage function, which implies the physical interpretation of stored energy in the system described by Eq. (1). Now, an equality with energy balance is introduced for the system. For any solution $(u(t), v(t))$, the following equality is obtained in the interval $[t_1, t_2]$:

$$\begin{aligned} &S_{\mathrm{v}}(u(t_2), v(t_2)) - S_{\mathrm{v}}(u(t_1), v(t_1)) \\ &\quad = \mu \int_{t_1}^{t_2} \left[1 - \beta u(\tau) - \gamma\{u(\tau)\}^2\right]\{v(\tau)\}^2 \mathrm{d}\tau \\ &\quad + B \int_{t_1}^{t_2} v(\tau)\cos\nu\tau \mathrm{d}\tau. \end{aligned} \tag{3}$$

On the right-hand side, the first term denotes dissipated energy during the interval, and the second term stands for energy supplied by the driving oscillatory force during the same interval. The equality (3) implies the energy conservation law of the system described by Eq. (1). The equality is called *an energy balance relation* for the system.

2.2. PLL Equation

The PLL equation [10] is represented by

$$\begin{cases} \dot{\phi} = y, \\ \dot{y} = -ky - \sin\phi + k\sigma + m\sqrt{k^2 + \Omega^2}\cos\Omega t, \end{cases} \tag{4}$$

where k and σ denote the fixed parameters. $k\sigma$ and $m\sqrt{k^2+\Omega^2}\cos\Omega t$ are the driving constant and oscillatory forces, respectively. The two variables (ϕ, y) belong to cylindrical phase space $S^1 \times \mathbb{R}$, because of the periodic restoring term $-\sin\phi$. A smooth function $S_{\rm p}(\phi, y)$ is defined as

$$S_{\rm p}(\phi, y) = \frac{y^2}{2} - \cos\phi. \tag{5}$$

$S_{\rm p}$ also implies the physical interpretation of stored energy in the system described by Eq. (4). Now, an equality with energy balance is introduced for the system. For any solution $(\phi(t), y(t))$, the following equality is obtained in the interval $[t_1, t_2]$:

$$\begin{aligned} &S_{\rm v}(\phi(t_2), y(t_2)) - S_{\rm v}(\phi(t_1), y(t_1)) \\ &= -k\int_{t_1}^{t_2}\{y(\tau)\}^2 \mathrm{d}\tau + k\sigma\int_{t_1}^{t_2} y(\tau)\mathrm{d}\tau \\ &\quad + m\sqrt{k^2+\Omega^2}\int_{t_1}^{t_2} y(\tau)\cos\Omega\tau \mathrm{d}\tau. \end{aligned} \tag{6}$$

On the right-hand side, the first term denotes dissipated energy during the interval. The second and third term stand for energy supplied by the driving constant and oscillatory forces during the same interval. The equality (6) is, therefore, regarded as an energy balance relation for the system described by Eq. (4).

3. Energy-Based Analysis [7]: Derivation of Response Curves for Supplied Energy

We extract response curves for harmonic amplitude and energy supplied by driving oscillatory force with their derivation from [7].

3.1. Van der Pol Equation

Figure 1 shows response curves for harmonic amplitude and supplied energy in Eq. (1). The parameter setting here is the same as in the references [2, 7]:

$$\mu = 0.15, \quad \beta = \gamma = \frac{4}{3}. \tag{7}$$

The frequency ν_0 of stable libration is almost unity. The harmonic amplitude derives from the following approximation of solution $u(t)$:

$$u(t) = A_{\rm v1}(t)\cos\{\nu t + \varphi_{\rm v1}(t)\}. \tag{8}$$

$A_{\rm v1}$ corresponds to the harmonic amplitude and $\varphi_{\rm v1}$ the phase difference between $u(t)$ and the driving oscillatory force $B\cos\nu t$. The solution (8) leads to an averaged equation with variables $A_{\rm v1}$ and $\varphi_{\rm v1}$. The derivation of the supplied energy depends on the energy balance relation (3). By substituting $u(\tau)$ and $v(\tau) = -\nu A_{\rm v1}(\tau)\sin\{\nu\tau + \varphi_{\rm v1}(\tau)\}$ into the equality (3) with the period $2\pi/\nu$, the following energy

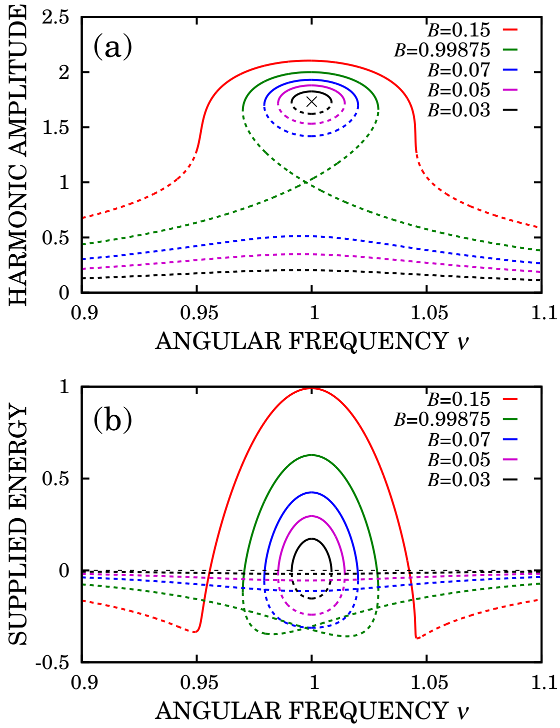

Figure 1: Response curves for (a) harmonic amplitude and (b) supplied energy in van der Pol equation. The figures are extracted from [7]. *Solid* lines denote the response curves for *stable* solutions, and *broken* lines for *unstable* solutions. The symbol $\times$ in (a) corresponds to the value of harmonic amplitude at the parameter set $(\nu, B) = (\nu_0, 0)$.

balance relation is derived for the system described by the averaged equation:

$$0 = \mu\pi\nu A_{\rm v1}^2\left(1 - \gamma\frac{A_{\rm v1}^2}{4}\right) - \pi A_{\rm v1}B\sin\varphi_{\rm v1}. \tag{9}$$

On the right-hand side, the first term denotes dissipated energy during the period, and the second term stands for energy supplied by the driving oscillatory force during the same period. The equality (9) makes it possible to derive the response curves for supplied energy in Fig. 1(b).

The feature of response curves in Fig. 1 is explained below. Each of the response curves has a maximum value for stable solutions near the frequency ν_0. The harmonic amplitude determines the sign of supplied energy. The reason is that the driving oscillatory force does positive work to the system under any solution with larger amplitude than the stable libration. Furthermore, the harmonic amplitude for stable solutions is larger than for coexisting unstable solutions at each parameter set (ν, B).

3.2. PLL Equation

Figure 2 shows response curves for harmonic amplitude and energy supplied by driving oscillatory force in Eq. (4). The parameters are set as the same as in the references [7, 11]:

$$k = 0.56, \quad \sigma = 1.7. \tag{10}$$

The frequency Ω_0 of stable rotation is about 1.6. The harmonic amplitude derives from the following approximation of solution $\phi(t)$:

$$\begin{cases} \phi(t) = \Omega t + x(t), \quad x(t) \equiv x\left(t + \dfrac{2\pi}{\Omega}\right), \\ x(t) = \dfrac{A_{p0}}{2} + A_{p1}(t)\cos\{\Omega t + \varphi_{p1}(t)\}. \end{cases} \tag{11}$$

A_{p0} corresponds to the dc component, A_{p1} the harmonic amplitude, and φ_{p1} the phase difference between $x(t)$ and the driving oscillatory force $m\sqrt{k^2+\Omega^2}\cos\Omega t$. The solution (11) leads to an averaged equation with variables A_{p1} and φ_{p1}. The derivation of response curve is also based on the energy balance relation (3). By substituting $\phi(\tau)$ and $y(\tau) = \Omega - \Omega A_{p1}(\tau)\sin\{\Omega\tau + \varphi_{p1}(\tau)\}$ into the equality (6) with the period $2\pi/\Omega$, the following energy balance relation is derived for the system described by the averaged equation:

$$0 = -\pi k\Omega(2+A_{p1}^2) + 2\pi k\sigma - \pi m\sqrt{k^2+\Omega^2}A_{p1}\sin\varphi_{p1}. \tag{12}$$

On the right-hand side, the first term denotes dissipated energy during the period. The second term stands for energy supplied by the driving constant force during the period, and the third term represents energy supplied by the driving oscillatory force during the same period. The averaged equation and the equality (12) make it possible to derive the response curves in Fig. 2. Numerical integration discriminates the stability.

The following depicts feature of response curves in Fig. 2. The supplied energy becomes zero for stable solutions near at minimum harmonic amplitude. Furthermore, the harmonic amplitude for stable solutions is smaller than for coexisting unstable solutions at each parameter set (Ω, m).

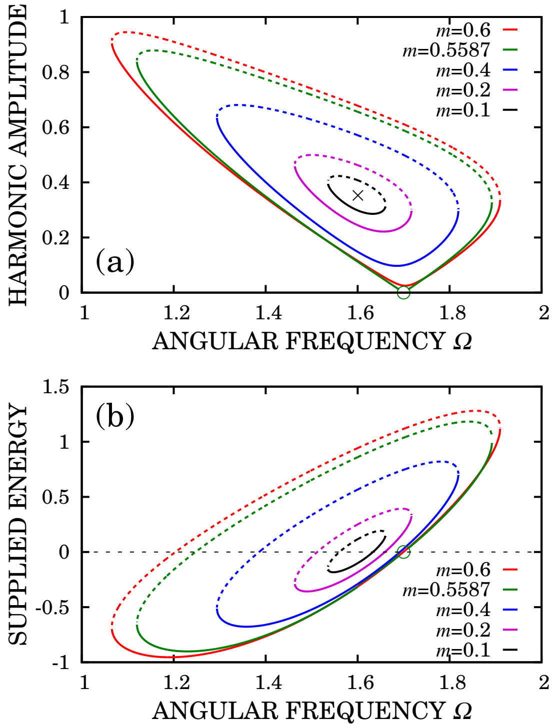

Figure 2: Response curves for (a) harmonic amplitude and (b) energy supplied by driving oscillatory force in PLL equation. The figures are extracted from [7]. *Solid* lines denote the response curves for *stable* solutions, and *broken* lines for *unstable* solutions. The symbol $\times$ in (a) represents the value of harmonic amplitude at a parameter set $(\Omega, m) = (\Omega_0, 0)$. In addition, the symbol $\odot$ corresponds to the point at which harmonic amplitude A_{p1} becomes zero.

4. An Analysis Based on Dissipated Power

In this section, we investigate the stability through the dissipated power in van der Pol and PLL equations.

4.1. Van der Pol Equation

For van der Pol equation, dissipated power is now defined. Eq. (1) is rewritten as

$$\begin{cases} \dot{w} = u, \\ -\dot{u} = w - \mu\left(u - \beta\dfrac{u^2}{2} - \gamma\dfrac{u^3}{3}\right) - \dfrac{B}{\nu}\sin\nu t. \end{cases} \tag{13}$$

Then $\mathcal{P}_v$, which is called the mixed-potential [12], is defined by

$$\mathcal{P}_v(w,u,t) = wu - \mu\left(\frac{u^2}{2} - \beta\frac{u^3}{6} - \gamma\frac{u^4}{12}\right) - \frac{B}{\nu}u\sin\nu t. \tag{14}$$

Time averaging of the mixed-potential corresponds to the averaged potential proposed by Kuramitsu *et al* [5, 6]. Here, the term of the averaged potential is addressed with respect to dissipation in Eq.(1). The term, denoted by P_v, is as follows:

$$P_v = -\mu\frac{A_{v1}^2}{4}\left(1 - \gamma\frac{A_{v1}^2}{8}\right). \tag{15}$$

The term P_v is related to dissipation in Eq. (1) and has a dimension of power in nonlinear circuits literature [5, 12]. It is, therefore, regarded as dissipated power. Fig. 3(a) shows response curves for dissipated power under the parameter setting (7) in Eq. (1). The dissipated power for stable solutions is smaller than for coexisting unstable solutions at each parameter set (ν, B).

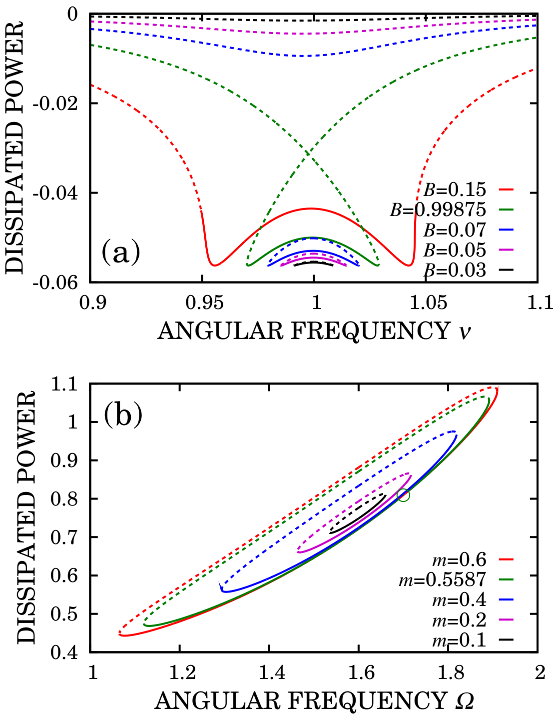

Figure 3: Response curves for dissipated power in (a) van der Pol equation and in (b) PLL equation. *Solid* lines denote the response curves for *stable* solutions, and *broken* lines for *unstable* solutions. The symbol ⊙ in (b) corresponds to the point at which harmonic amplitude $A_{\mathrm{p}1}$ becomes zero.

4.2. PLL Equation

For PLL equation, dissipated power is similarly defined. Eq. (4) is rewritten in [13] as

$$\begin{cases} \cos\phi\,\dot{\phi} = y\cos\phi, \\ \quad -\dot{y} = \sin\phi + ky - k\sigma - m\sqrt{k^2+\Omega^2}\cos\Omega t. \end{cases} \tag{16}$$

Then, the mixed-potential $\mathcal{P}_{\mathrm{p}}$ is defined by

$$\mathcal{P}_{\mathrm{p}} = y\sin\phi + \frac{1}{2}ky^2 - k\sigma y - m\sqrt{k^2+\Omega^2}\,y\cos\Omega t. \tag{17}$$

Here, in the same manner as van der Pol equation, the dissipated power P_{p} for the averaged system is addressed:

$$P_{\mathrm{p}} = \frac{1}{4}k\Omega^2(2 + A_{\mathrm{p}1}^2). \tag{18}$$

Fig. 3(b) shows response curves for the dissipated power under the parameter setting (10). The dissipated power for stable solutions is smaller than for coexisting unstable solutions at each parameter set (Ω, m).

5. Concluding Remark

In this paper, dissipated power was applied for analyzing the frequency entrainment described by van der Pol and PLL equations. The power corresponds to the averaged potential for dissipation elements in nonlinear circuits, which was proposed by Kuramitsu *et al.* The contribution of this paper is that the dissipated power for stable solutions becomes minimum value than for coexisting unstable solutions in the equations. In other words, the dissipated power shows the same feature related to stability for the frequency entrainment with different types of limit cycle. The relation between the power and stability corresponds to the important statement on synchronization made by Kuramitsu *et al* [5, 6]: averaged potential becomes a minimum value at stable equilibrium point.

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