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Optimal Networks, Congestion and Braess' Paradox

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Abstract—We present a brief description of the topological properties of some optimal networks. The topology depends on the network's load. For low loads, the networks are star–like, there is an intermediate regime where the networks have a narrow degree distribution, and for high loads the networks tend to be regular (or almost regular). The addition of extra links to the optimal network reduces its overall performance. This deterioration is small if the network structure has a narrow degree distribution.

1. Introduction

An approach to built optimal networks is to considered many of its characteristics fixed (e.g. routing mechanism, network load, traffic characteristics) and then obtain a network connectivity that produces the most efficient network [1, 2, 3, 4]. In a packet network the routing mechanism is one of the bounding elements between the network's topology and the traffic that it carries. This interaction is particularly important when some of the network's nodes are congested [5, 6, 7, 8, 9]. In this case, the routing mechanism "searches" the network topology for alternative routes that avoid the congested nodes. If we consider that the routing mechanism is given, then we want to connect the network such that the travelling time of a packet is as small as possible. If the network is made of links and nodes which contain queues, a packet will be delayed due to the transmission time and by the time spent on the queues. Here we consider that the transmission time is fixed so if the network is heavily used the main contributor to delay is due to queueing.

Consider the journey time between two nodes in the network where there is at least two shortest paths between the nodes. The delay time $\bar{\tau}_{s,d}$ is the average time that elapses between the creation of a packet at source s to the arrival to its destination d. If the network has $\mathcal{S}$ nodes, which are sources and sinks of traffic, then the average packet delay $\bar{\tau} = 1/\mathcal{S}(\mathcal{S}-1)\sum_{s,d}\bar{\tau}_{s,d}$, is an indicator of network's performance. If the traffic load presented to the network is low and the queues on the nodes are empty, then, to first approximation, the average delay is proportional to the average number of nodes that the packets visit. For low load, the average delay is $\bar{\tau} \approx \bar{\ell}$, where $\bar{\ell} = 1/\mathcal{S}(\mathcal{S}-1)\sum_{s,d}\ell_{s,d}$, is the average of all the shortest paths, where $\ell_{s,d}$ is the shortest path from source s to destination d.

The journey time of two shortest paths with the same length can be very different due to the different patterns of usage of the routes. Some nodes are more "prominent" because they are highly used when transferring packet-data. The greater number of shortest paths which a node contributes in the delivery of information can be quantify using the betweenness centrality defined as [10].

$$C_B(w) = \sum_s \sum_{d\neq s} \frac{g(w; s, d)}{g(s, d)} \tag{1}$$

where $g(w; s, d)$ is the number of shortest paths between nodes s and d that pass through node w, $g(s, d)$ is the total number of shortest paths between s and d. If the packets on the network are distributed evenly through all the shortest paths then the normalise betweenness $\hat{C}_B(w) = C_B(w)/\sum_v C_B(v)$ gives the proportion of usage of node w. If each nodes generates a load of Λ packets, then average packet arrivals at node i is $\lambda_i = \Lambda\mathcal{S}\bar{\ell}\hat{C}_B(i) = \Lambda C_B(i)/(\mathcal{S}(\mathcal{S}-1))$ [9], where we used the relationship $\sum_{i=1}^{S} C_B(i) = \mathcal{S}(\mathcal{S}-1)\bar{\ell}$.

The critical load in terms of the betweenness is obtained using [5]

$$\frac{dN(t)}{dt} = \Lambda\mathcal{S} - \frac{N(t)}{\tau(t)}, \tag{2}$$

where $\Lambda\mathcal{S}$ is the average rate of packets generation per unit of time, $\bar{\tau}(t)$ is the time spent in the system, and $N(t)/\bar{\tau}(t)$ is the number of packets delivered per unit of time. The time spent in the system can be approximated by the average packet delay $\tau(t) \approx \bar{\tau} = 1/\mathcal{S}\sum_{i=1}^{S} T_i$, where T_i is the time spent in queue i plus the service time of the server. If the network is not congested from the steady state solution $dN/dt = 0$ gives

$$\bar{N} = \Lambda S\,\bar{\tau} = \Lambda\sum_{i=1}^{S} T_i = \sum_{i=1}^{S}\frac{\Lambda(S-1)}{\mu_i(1-S) - \Lambda C_B(i)}. \tag{3}$$

where we have assumed that the queues are M/M/1, with average arrivals λ_i and service rate μ_i.

If the load is low $\Lambda \approx 0$ then $\bar{N} \approx \Lambda S\,\bar{\ell}$. For high loads the majority of the packets of the network are on the busiest queue. If m labels the busiest queue Q_m then $\bar{N} \approx \bar{Q}_m$, at the congestion point $\bar{N} \to \infty$ and the critical load is [9]

$$\Lambda_c = \frac{\mu_m(S-1)}{C_B(m)}. \tag{4}$$

2. Optimal Networks

The networks is constructed by fixing the load $\Lambda \in (0, \Lambda_c)$, the number of node $\mathcal{S}$ and the number of links $\mathcal{L}$, and then reconnecting the network until $\bar{N}$ is minimal [1] (using simulated annealing). For low loads, the optimal solution is a star–like network. In this case the network contains at least one node with a high degree of links that plays a central role in the exchange of information between the rest of the nodes. In the case that the load Λ is near congestion, the optimal solution is a *homogeneous–isotropic* network [1]. A network is homogeneous–isotropic if the degree (number of links) of the nodes are narrowly distributed around the average $\mathcal{L}/\mathcal{S}$, where $\mathcal{L}$ is the total number of links in the network.

The change of network connectivity as a function of the load is measured using the polarisation defined as [1]

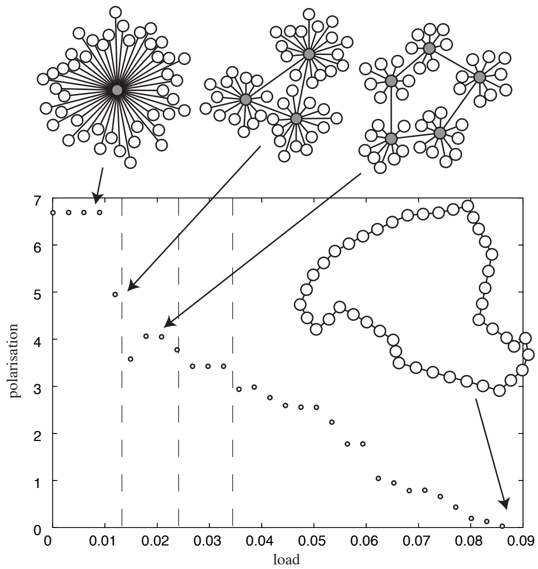
$$\pi' = \frac{\bar{C}_B^* - \bar{C}_B}{\bar{C}_B} = \frac{\bar{\ell}^*}{\bar{\ell}} - 1 \qquad (5)$$

where $\bar{C}_B^*$ is the betweenness of the homogeneous–isotropic network with the largest congestion load Λ_c^* and $\bar{C}_B$ is the average betweenness of the network. An abrupt change on the polarisation means an abrupt change on the average shortest–path of the network. For high loads $\pi' \to 0$, for low loads π' is relatively large.

Figures 1 and 2 show the optimal network topologies and the change in the polarisation with the load. Figure 1 shows the case that $\mathcal{S} = \mathcal{L}$ and figure 2 the case $\mathcal{S} = 2\mathcal{L}$, the service rate is $\mu_i = 1, \forall i$. In both cases for low load the networks structure is star–like, where one or more center nodes organise the structure of the network. For high loads, all the network's node have the same number of links. i.e the network is represented by a regular graph.

3. Low load

If the network has $\mathcal{S}$ nodes, $\mathcal{L} = \mathcal{S}$ links and the same service rate for all queues $\mu_i = \mu$ the optimisation algorithm produces star–like networks for low loads and a ring network for high loads. The change of the polarisation as the load is increased is shown in figure 1. As the load increases the topology changes from a 1–star–like network to a 3–star–like to a 5–star–like network and so on until the network becomes a ring [2]. This sequence of optimal networks can be obtained analytically. In the star–like networks, we are going to call a "centre" the node(s) with a high degree and "rays" the nodes that connect to the centres and they are not centres themselves. If we remove all the rays node from the network, what is left we call it the skeleton network (grey nodes in figure 1). The betweenness of the nodes are $C_B(ray) = \mathcal{S} - 1$ and $C_B(centre) = (C_B(Skeleton)\mathcal{S}^2 - \mathcal{S}c + \mathcal{S}^2 c + S^2 - \mathcal{S}c^2)/c^2$ where c is the number of stars and $C_B(Skeleton)$ is the betweenness of the nodes in the skeleton network. The total



The polarisation of the network decreases as the load is increased. For low load $\lambda = 0.01$ network is a (a) 1–star network as the load increases it changes to a (b) 3–star network ($\lambda = 0.015$), to a (c) 5–star network. The transitions are marked with a vertical dashed line. As the load increases the network changes topologies to networks with more stars. For values greater than $\Lambda \approx 0.1$ (d) the network has the topology of a ring. The network has 45 nodes and 45 links.

number of packets in the network is

$$\bar{N}_{starlike} = (\mathcal{S} - c)\bar{Q}_{ray} + c\bar{Q}_{centre} \qquad (6)$$

where for M/M/1 queues, $Q_\star = \rho_\star/(1 - \rho_\star)$, $\rho_\star = \lambda_\star/\mu_\star$ and $\lambda_\star = \Lambda C_B(\star)/(\mathcal{S} - 1)$. Evaluating $\Delta\bar{N}_{i,j}(\Lambda) = \bar{N}_{i-star}(\Lambda) - \bar{N}_{j-star}(\Lambda)$ is not difficult to prove that the transition 1-, 3-, 5- ... starlike are the optimal networks as the load increases.

4. Medium load

Figure 2 shows a typical transition from star networks to regular networks. There is an interval where the networks are not star–like or regular networks (the interval $0.05 < \Lambda < 0.014$ marked on the figure). In this interval the degree of the node fluctuates closely to the average degree $2\mathcal{L}/\mathcal{S} = 4$ and they have a relatively large girth (the length of its shortest circuit). This is a characteristic also found in the synchronisation of dynamical systems on a network (entangled networks) [4] and in networks without communities [11].

5. High load

For high load and if $\mathcal{L}/\mathcal{S} = p$ with p and integer, the nodes of the network all have the same number of links, i.e all the nodes have the same degree. To classify these regular networks we evaluated the node–connectivity and link–connectivity. The node–connectivity κ is the minimum number of nodes needed to remove to obtain a disconnected network. The link–connectivity κ' is the minimum number of links needed to remove to obtain a disconnected network. If d_{min} is the smallest degree that appears on the network, then for any graph $\kappa \leq \kappa' \leq d_{min}$. An *optimally connected* network satisfy $\kappa = \kappa' = d$ [3]. These optimally connected networks are considered very robust as all the nodes look the same.

In figure 2 (a) for $\Lambda > 0.014$, the networks are all optimally connected. However, the connectivity of the networks changes with the load, in figure 2 (a) we have marked the change of girth on these networks. The different girth of the optimal networks can have an important effect on the network's behaviour, particular if there is a link failure [3], in this case the load in the links will increase if the *girth* is large.

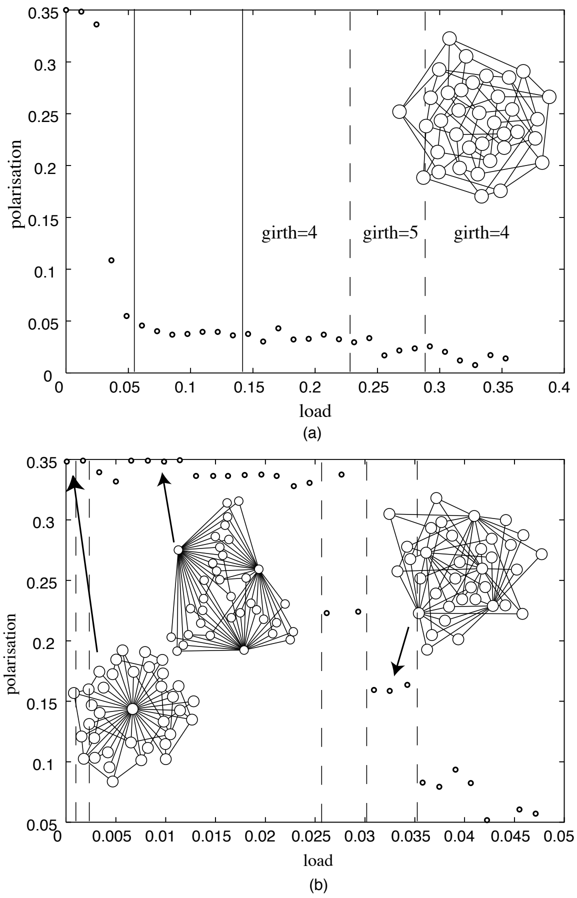


Figure 2: (a) Polarisation for a network with 36 nodes and 72 links. From $0 < \Lambda < 0.05$ the networks are star–like, from $0.05 < \Lambda < 0.145$ the degree of nodes fluctuate closely to the average degree $\mathcal{L}/\mathcal{S}$. From $0.145 < \Lambda < 0.38$ the optimal solution are regular networks. (b) Change of the polarisation for low loads.

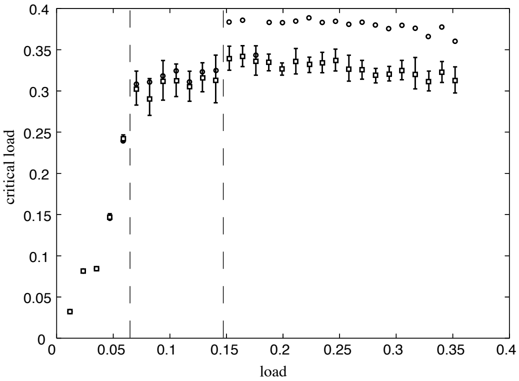
Remarks: a network where the betweenness is the same for all nodes does not implies that the network is optimal. A network described by a regular graph does not necessarily has a $C_B(i)$ which is the same for all the nodes.

6. Braess' paradox

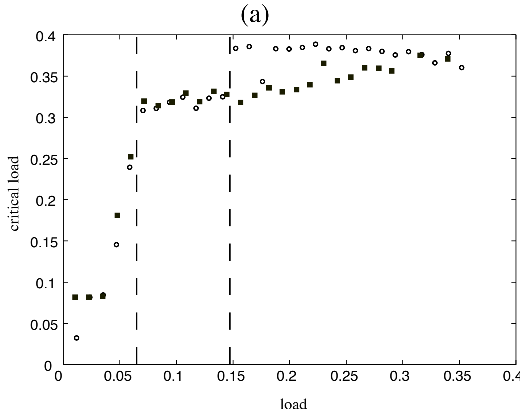
Figure 3(a) shows the the critical load Λ_c (eq. 4) versus the load when one random link is added to the optimal networks shown in figure 2. For the star–like networks ($\Lambda < 0.07$) and the networks that have a degree distribution narrowly cantered at $\mathcal{L}/S$ ($0.07 < \Lambda < 0.14$), the addition of one link changes the critical load by a small amount. This is not the case for regular symmetric networks ($\Lambda > 0.14$) which congest more readily. Figure 3(b) shows the onset on congestion when one link is added to the network and then the optimisation procedure is used to obtain a new optimal network. As the case of the addition of a random link, the network that has one extra link tends to congest more readily relative to the symmetric networks. The behaviour shown in figure 3(b) is an example of Braess' paradox [12]; increasing the capacity of the network, by adding an extra link, reduces the overall performance of the network (congestion happens at lower values). Braess' paradox is an undesirable property in a network and its avoidance is an active field of research (see for example [13, 14, 5, 14]). From the numerical experiments presented here it seems that it is possible to diminish the effect of Braess' paradox by topology considerations only. If the optimal network is almost regular (a narrow degree distribution around $\mathcal{L}/S$), then the addition of an extra link has very little effect on the performance of the whole network.

7. Conclusions

For low load, the optimal network structure is star–like. As the load increases there is a transition to a homogeneous decentralised structure. The transition from star–like to decentralised structure is sharp and it follows a pattern. Star–like networks change from having a small number of centres, to greater number of centres and then finally, decentralise. For star–like networks the transition between structures happens well before any of the nodes gets congested. The decentralised structures also change as the load increases. If the number of links is a multiple of the number of nodes, the decentralised structure changes from a homogeneous network to a regular symmetric network. For high loads not all possible regular symmetric networks are optimal, so symmetry is necessary but not sufficient for the design of optimal networks. The regular–symmetric networks are robust in the sense that degree–wise all nodes are the same. No node is more prominent than the others so there is no added value target for a network attack. The addition of one extra link to a regular network will not increase the vulnerability to an attack. However, the addition of one link to the network can reduce the overall network performance. Networks that have a narrow degree distribution (entangled networks) are robust to attacks and to deterioration of overall performance due to the addition of extra links.



(a)



(b)

Figure 3: (a)The open circles are the values of the critical load for the optimal network. The squares plus error bars is the average critical load obtained from an ensemble of 100 networks which were created by adding at random one extra link to the optimal network. (b)The circles show the value of the critical load for the optimal networks with $\mathcal{S}$ nodes and $\mathcal{L}$ links. The solid squares are the critical load of the optimal network with $\mathcal{S}$ nodes and $\mathcal{L}+1$ links. In both cases the original network has $\mathcal{S}=36$, $\mathcal{L}=72$ and $\mu_i=1$.

Acknowledgments

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