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Spatio-temporal Oscillations by Wave Bifurcation

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Abstract

We shall describe the mechanism for the various type of spatio-temporal oscillations by finding a sort of organizing center in the sense of bifurcation. Wave bifurcation is one of the key idea to understand such a structure. We shall show that the wave instability can be seen in 3-component reaction-diffusion system under certain conditions. The interaction between the wave and Turing bifurcations can be also seen universally. We shall introduce the bifurcation analysis about these critical points and discuss how much spatio-temporal oscillations we can explain by the wave bifurcation.

Temporal oscillations are commonly observed in a variety of phenomena such as BZ chemical reaction. Consider now a coupled system of these oscillatory dynamics so that it exhibits spatio-temporal oscillations. Motivated by chemical reactions we consider spatial diffusion effects as the coupling and study how these systems exhibit spatio-temporal oscillations. Actually we study the reaction diffusion(RD) system with oscillatory dynamics. One can observe a variety of complex activities there, compared to the RD systems with excitable dynamics or the so-called Turing dynamics. We shall concentrate on the bifurcations from the rest state first and then try to understand more global view of behaviors.

Oscillating motion appears as the Hopf bifurcation for a certain parameter. Now, we need to distinguish the following two situations. One possibility is that the Hopf bifurcation to a spatially uniform oscillation takes place earlier than that to spatially non-trivial oscillations. There could be another possibility, namely, the Hopf bifurcation to the non-zero wave number mode might take place earlier than the uniform mode. We call the latter case "wave bifurcation". (See 2), 4) and 6).)

It is easy to show that there is no wave bifurcation in 2-component RD systems as we shall see later. Then, how we can understand the various spatio-temporal oscillations which we can actually observe in both actual and numerical experiments? One may imagine the case that the branch of the spatially non-trivial oscillation may recover its stability far from an equilibrium even if the corresponding Hopf bifurcation takes place later than that of the uniform mode. Or branches of the spatio-temporal oscillations do not necessary connect to the rest state. However, we shall not study these spacial cases here since we think the wave bifurcation is the first key to understand the spatio-temporal oscillations. In fact we will show that 3-component RD systems can produce the wave bifurcation. Moreover we will introduce the bifurcation analysis for the solutions resulting from this wave bifurcation.

Let us start with the 2-dimensional ODE which has a limit cycle coursed by the Hopf bifurcation:

$$\begin{aligned} \dot{u} = f(u,v) &:= p(u-v) - u^3, \\ \dot{v} = g(u,v) &:= qu - v. \end{aligned}$$

Here, we fix a constant $q > 1$ and consider p as the bifurcation parameter. In fact, $p = 1$ is the supercritical Hopf bifurcation point. Now, consider the RD system:

$$\begin{aligned} u_t &= D_1 \Delta u + f(u,v), \\ v_t &= D_2 \Delta v + g(u,v). \end{aligned}$$

The linearized stability about its trivial uniform solution is given by the following matrix for each wave number k:

$$A_k = \begin{pmatrix} p - D_1 k^2 & -p \\ q & -1 - D_2 k^2 \end{pmatrix}.$$

Since $\mathrm{Trace} A_k = p - 1 - (D_1 + D_2)k^2$ the critical set for the Hopf bifurcation is given by

$$C_H := \{(k,p); \mathrm{Trace} A_k = 0\} = \{p = 1 + (D_1 + D_2)k^2\}$$

if the imaginary part of the eigenvalues is not zero. The critical mode for $k = 0$ may be destabilized at $p = 1$ earlier than the other critical mode for $k \neq 0$. Therefore, spatially non-uniform oscillation may bifurcate later than the uniform oscillation and, as a result, only the uniform oscillation can be observable. This means that we do not have wave bifurcation in 2-component RD systems. It should be mentioned here that we did not say anything about static bifurcations. In fact, a static bifurcation can take place earlier than the uniform Hopf mode by taking the ratio of two diffusion coefficients appropriately.

Next consider the following 3-component RD system where the third component has negative feedback effect to the oscillator.

$$\begin{aligned} u_t &= D_1\Delta u + f(u,v) - sw \\ v_t &= D_2\Delta v + g(u,v) \\ \tau w_t &= D_3\Delta w + u - w \end{aligned}$$

If the time constant τ for the third component w is sufficiently small the wave bifurcation occurs by taking the feedback constant s appropriately. In fact take $\tau = 0$ and we have formally

$$D_3\Delta w + u - w = 0.$$

By solving this limit equation as

$$w = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-|x-\eta|/D_3} u(\eta)d\eta$$

the system can be reduced to the following 2-component RD system with a global coupling.

$$\begin{aligned} u_t &= D_1\Delta u + f(u,v) \quad -s\int e^{-\frac{|x-\eta|}{D_3}} u(\eta)d\eta \\ v_t &= D_2\Delta v + g(u,v) \end{aligned}$$

The linearized eigenvalue problem is now given by

$$\tilde{A}_k = \begin{pmatrix} p - D_1k^2 - \frac{s}{1+D_3k^2} & -p \\ q & -1 - D_2k^2 \end{pmatrix}.$$

By calculating Trace$\tilde{A}_k$ we can conclude that the wave bifurcation occurs when

$$D_3 > \frac{D_1 + D_2}{s}.$$

We shall introduce the bifurcation analysis results about this wave bifurcation and moreover the mixed bifurcation between the wave and Turing bifurcations.

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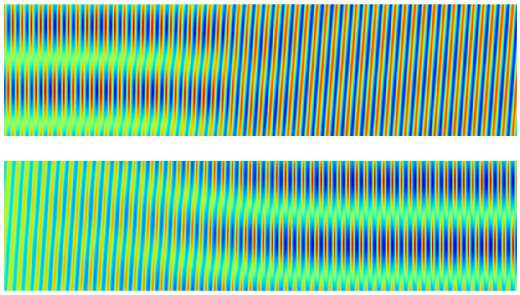

Figure 1: Rotating and standing oscillations of 3-component RDs with different nonlinear terms. Vertical and horizontal axis correspond to space(periodic) and time (from left to right).

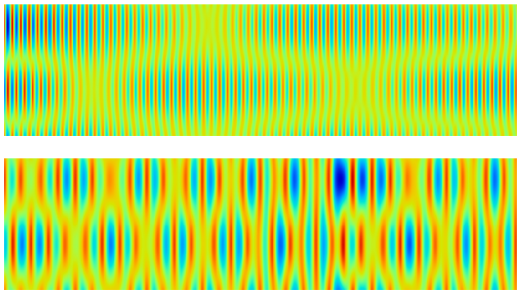

Figure 2: Mixed mode oscillations between standing and uniform oscillations in a 3-component RD.

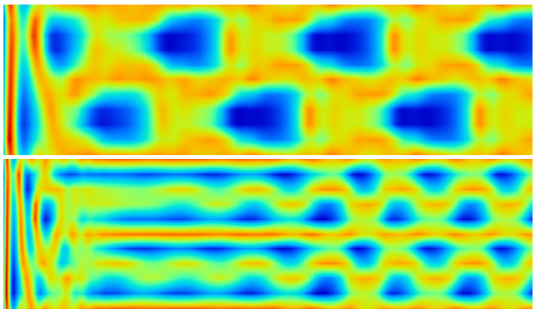

Figure 3: Mixed mode oscillations between wave and Turing bifurcations in 3-component RDs.

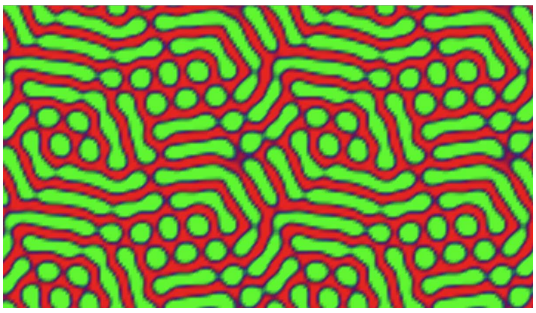

Figure 4: Wave-Turing bifurcation can produce a variety of 2D Turing patterns.