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Numerical verification of bifurcating solutions with multi-peaks for 3-dimensional Rayleigh-Bénard convection

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Abstract—This is an extension of our previous work [3]. Numerical verification of the existence of nontrivial steady state solutions with multi-peaks for 3 dimensional Rayleigh–Bénard convection is studied based on the fixed point theorem using a Newton like operator with the spectral Galerkin method.

1. Introduction

The steady state bifurcation equations for the perturbation $(\mathbf{u}, \theta, p)$ to the equilibrium of Rayleigh–Bénard convection are given by [3]

$$-\Delta \mathbf{u} + \frac{1}{\mathcal{P}}(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \mathcal{R}\theta\mathbf{e}_3 = \mathbf{0}, \quad (1a)$$

$$\nabla\cdot\mathbf{u} = 0, \quad (1b)$$

$$-\Delta\theta + (\mathbf{u}\cdot\nabla)\theta - u_3 = 0 \quad (1c)$$

where $\mathcal{R}, \mathcal{P}$ are Rayleigh and Prandtl numbers in the media between two parallel plates $(\mathbb{R}\times\mathbb{R}\times[0,\pi])$.

Assume parity conditions [6] on $\Omega \equiv [0, \frac{2\pi}{a}] \times [0, \frac{2\pi}{b}] \times [0, \pi]$ for given wave numbers $a, b > 0$ with the measure $|\Omega| \equiv \frac{4\pi^3}{ab}$. Then the solution of (1) can be represented by Fourier series [3]: $\mathbf{u} = \sum_{\alpha\neq\mathbf{0}}[u_\alpha\phi_1^\alpha, v_\alpha\phi_2^\alpha, w_\alpha\phi_3^\alpha]$, $\theta = \sum_{\alpha_3\neq 0}\theta_\alpha\phi_3^\alpha$, $p = \sum_{\alpha\neq\mathbf{0}} p_\alpha\phi_4^\alpha$, where $\alpha \equiv (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_0^3$ is the multi-index of non–negative integers, and $(u_\alpha, v_\alpha, w_\alpha, \theta_\alpha, p_\alpha)$ are coefficients of $(\mathbf{u}, \theta, p)$ with respect to the base functions ϕ_i^α defined by,

$$\phi_1^\alpha(x,y,z) \equiv K_\alpha \sin(a\alpha_1 x)\cos(b\alpha_2 y)\cos(\alpha_3 z),$$
$$\phi_2^\alpha(x,y,z) \equiv K_\alpha \cos(a\alpha_1 x)\sin(b\alpha_2 y)\cos(\alpha_3 z),$$
$$\phi_3^\alpha(x,y,z) \equiv K_\alpha \cos(a\alpha_1 x)\cos(b\alpha_2 y)\sin(\alpha_3 z),$$
$$\phi_4^\alpha(x,y,z) \equiv K_\alpha \cos(a\alpha_1 x)\cos(b\alpha_2 y)\cos(\alpha_3 z).$$

Here $K_\alpha \equiv \sqrt{(2-\delta_{0\alpha_1})(2-\delta_{0\alpha_2})(2-\delta_{0\alpha_3})/|\Omega|}$ is the normalization factor with respect to $L^2(\Omega)$ inner product $\langle\cdot,\cdot\rangle$ and δ_{ij} is the Kronecker delta on i, j.

Norms for $\mathbf{u}$, θ and p are: $\|\mathbf{u}\|_0^2 = \sum_{\alpha\neq\mathbf{0}}\{u_\alpha^2 + v_\alpha^2 + w_\alpha^2\}$, $\|\nabla\mathbf{u}\|_0^2 = \sum_{\alpha\neq\mathbf{0}}\{u_\alpha^2 + v_\alpha^2 + w_\alpha^2\}A_\alpha^2$, $\left\|\nabla^2\mathbf{u}\right\|_0^2 = \sum_{\alpha\neq\mathbf{0}}\{u_\alpha^2 + v_\alpha^2 + w_\alpha^2\}A_\alpha^4$, $\|\theta\|_0^2 = \sum_{\alpha_3\neq 0}\theta_\alpha^2$, $\|\nabla\theta\|_0^2 = \sum_{\alpha_3\neq 0}\theta_\alpha^2 A_\alpha^2$, $\left\|\nabla^2\theta\right\|_0^2 = \sum_{\alpha_3\neq 0}\theta_\alpha^2 A_\alpha^4$, $\|p\|_0^2 = \sum_{\alpha\neq\mathbf{0}} p_\alpha^2$, $\|\nabla p\|_0^2 = \sum_{\alpha\neq\mathbf{0}} p_\alpha^2 A_\alpha^2$, $\left\|\nabla^2 p\right\|_0^2 = \sum_{\alpha\neq\mathbf{0}} p_\alpha^2 A_\alpha^4$, $\|\Delta\mathbf{u}\|_0 = \left\|\nabla^2\mathbf{u}\right\|_0$, $\|\Delta\theta\|_0 = \left\|\nabla^2\theta\right\|_0$, $\|\Delta p\|_0 = \left\|\nabla^2 p\right\|_0$, where $A_\alpha \equiv \sqrt{(a\alpha_1)^2 + (b\alpha_2)^2 + \alpha_3^2}$.

Divergence free orthogonal base functions are:

$$\Phi^\alpha \equiv \left[-\frac{a\alpha_1\alpha_3}{A_\alpha B_\alpha}\phi_1^\alpha, -\frac{b\alpha_2\alpha_3}{A_\alpha B_\alpha}\phi_2^\alpha, \frac{B_\alpha}{A_\alpha}\phi_3^\alpha\right], \ \alpha\in I_1,$$

$$\Psi^\alpha \equiv \left[\frac{b\alpha_2}{B_\alpha}\phi_1^\alpha, -\frac{a\alpha_1}{B_\alpha}\phi_2^\alpha, 0\right], \ \alpha\in I_2,$$

where $B_\alpha \equiv \sqrt{(a\alpha_1)^2 + (b\alpha_2)^2}$ and indices subsets are $I_1 \equiv [1,0,1]+\mathbb{Z}_0^3 \cup [0,1,1]+\mathbb{Z}_0^3$, $I_2 \equiv [1,1,0]+\mathbb{Z}_0^3$. Set $I_0 = I_1 \cup I_2$ and then define function spaces V and W as follows:

$$V = \left\{\mathbf{u} = \sum_{\alpha\in I_0}\{\xi_\alpha\Phi^\alpha + \eta_\alpha\Psi^\alpha\} \,:\, \|\Delta\mathbf{u}\|_0 < \infty\right\} \subset H^2(\Omega)^3,$$

$$W = \left\{\theta = \sum_{\alpha\in I_3}\theta_\alpha\phi_3^\alpha \,:\, \|\Delta\theta\|_0 < \infty\right\} \subset H^2(\Omega),$$

where $I_3 \equiv [0,0,1] + \mathbb{Z}_0^3$. Note $\|\mathbf{u}\|_0^2 = \sum_{\alpha\in I_0}\{\xi_\alpha^2 + \eta_\alpha^2\}$, $\|\nabla\mathbf{u}\|_0^2 = \sum_{\alpha\in I_0}\{\xi_\alpha^2+\eta_\alpha^2\}A_\alpha^2$, $\|\Delta\mathbf{u}\|_0^2 = \sum_{\alpha\in I_0}\{\xi_\alpha^2+\eta_\alpha^2\}A_\alpha^4$ for all $\mathbf{u}\in V$, and $\|\theta\|_0^2 = \sum_{\alpha\in I_3}\theta_\alpha^2$, $\|\nabla\theta\|_0^2 = \sum_{\alpha\in I_3}\theta_\alpha^2 A_\alpha^2$, $\|\Delta\theta\|_0^2 = \sum_{\alpha\in I_3}\theta_\alpha^2 A_\alpha^4$ for all $\theta\in W$.

2. A priori error estimates

For a fixed positive integer N, define finite dimensional subspaces $V_N = \left\{\mathbf{u}\in V \,:\, \xi_\alpha = \eta_\alpha = 0, \ ^\forall\alpha\in I_0, |\alpha| > N\right\}$, $W_N = \left\{\theta\in W \,:\, \theta_\alpha = 0, \ ^\forall\alpha\in I_3, |\alpha| > N\right\}$ where $|\alpha| \equiv \alpha_1 + \alpha_2 + \alpha_3$. And define projections $P_N : V \to V_N$ and $Q_N : W \to W_N$ as follows[7]:

$$\langle\nabla(\mathbf{u} - P_N\mathbf{u}), \nabla\mathbf{v}\rangle = 0, \quad ^\forall\mathbf{v}\in V_N, \quad (2a)$$

$$\langle\nabla(\theta - Q_N\theta), \nabla\vartheta\rangle = 0, \quad ^\forall\vartheta\in W_N. \quad (2b)$$

Due to orthogonal relations of base functions of V and W, these projections (2) are truncation operators:

$$P_N\mathbf{u} = \sum_{\alpha\in I_{0,N}}\{\xi_\alpha\Phi^\alpha + \eta_\alpha\Psi^\alpha\}, \quad Q_N\theta = \sum_{\alpha\in I_{3,N}}\theta_\alpha\phi_3^\alpha,$$

$$I_{0,N} \equiv \{\alpha\in I_0 \,:\, |\alpha|\le N\}, \quad I_{3,N} \equiv \{\alpha\in I_3 \,:\, |\alpha|\le N\}.$$

From these characterization of projections, we have (for proofs of Theorem 1, Lemma 2, Corollary 3, see [4])

Theorem 1 *For any* $(\mathbf{u},\theta) \in X$ *and* $(P_N\mathbf{u}, Q_N\theta) \in X_N$ *in (2), the following holds:*

$$\|\mathbf{u} - P_N\mathbf{u}\|_0 \le \frac{C_0^2}{(N+1)^2}\|\Delta\mathbf{u}\|_0, \tag{3a}$$
$$\|\nabla(\mathbf{u} - P_N\mathbf{u})\|_0 \le \frac{C_0}{N+1}\|\Delta\mathbf{u}\|_0, \tag{3b}$$
$$\|\theta - Q_N\theta\|_0 \le \frac{C_0^2}{(N+1)^2}\|\Delta\theta\|_0, \tag{3c}$$
$$\|\nabla(\theta - Q_N\theta)\|_0 \le \frac{C_0}{N+1}\|\Delta\theta\|_0, \tag{3d}$$

where $C_0 \equiv \sqrt{1 + a^{-2} + b^{-2}}$ *depends only on* Ω.

Lemma 2 *For any* $(\mathbf{u},\theta) \in X$, *it holds that*

$$\|\mathbf{u}\|_\infty \le \frac{\pi}{3}\sqrt{6 - \frac{2\pi^2}{5}}\, C_1\|\Delta\mathbf{u}\|_0,$$
$$\|\theta\|_\infty \le \frac{\pi}{3}\sqrt{6 - \frac{36\zeta(3)}{\pi^2} + \frac{\pi^2}{5}}\, C_1\|\Delta\theta\|_0,$$

where $C_1 \equiv C_0^2|\Omega|^{-\frac{1}{2}}$ *depends only on* Ω *and* $\zeta(s) \equiv \sum_{n=1}^{\infty}\frac{1}{n^s}$ *is the Riemann zeta function for* $s > 1$.

Corollary 3 *Under the same assumptions of Theorem 1, the following holds:*

$$\|\mathbf{u} - P_N\mathbf{u}\|_\infty < \frac{2C_1}{\sqrt{N}}\|\Delta\mathbf{u}\|_0,\ \|\theta - Q_N\theta\|_\infty < \frac{2C_1}{\sqrt{N}}\|\Delta\theta\|_0. \tag{4}$$

3. A fixed point formulation

The steady state solution of (1) satisfies

$$-\Delta\mathbf{u} + \nabla p = \mathbf{f}(\mathbf{u},\theta), \tag{5a}$$
$$\nabla\cdot\mathbf{u} = 0, \tag{5b}$$
$$-\Delta\theta = g(\mathbf{u},\theta), \tag{5c}$$

where the right hand sides of (5) are defined by

$$\mathbf{f}(\mathbf{u},\theta) = -\frac{1}{\mathcal{P}}(\mathbf{u}\cdot\nabla)\mathbf{u} + \mathcal{R}\theta\mathbf{e}_z, \quad g(\mathbf{u},\theta) = -(\mathbf{u}\cdot\nabla)\theta + w.$$

Define $F(\mathbf{u},\theta) \equiv (\mathbf{f}(\mathbf{u},\theta), g(\mathbf{u},\theta))$. We call the solution operator $\mathcal{S}$ for (5) as Stokes operator. Thus $(\mathbf{u},\theta) = \mathcal{S}F(\mathbf{u},\theta)$ means: for all $(\mathbf{v},\vartheta) \in X$

$$\langle \nabla\mathcal{S}F(\mathbf{u},\theta), \nabla(\mathbf{v},\vartheta)\rangle = \langle F(\mathbf{u},\theta), (\mathbf{v},\vartheta)\rangle. \tag{6}$$

Note that $\mathcal{S}^{-1}(\mathbf{u},\theta) = (-\Delta\mathbf{u} + \nabla p, -\Delta\theta)$ with an associated pressure $p = p(\mathbf{u},\theta)$.

Usually, we use Newton's method (see [5]) to get an approximate solution $(\mathbf{u}_N,\theta_N) \in X_N$ of (5) and get an approximate pressure p_N from $\nabla p_N \equiv \mathbf{f}_N(\mathbf{u}_N,\theta_N) + \Delta\mathbf{u}_N$, where $\mathbf{f}_N$ is the truncation up to $|\alpha| \le N$ of the expansion of $\mathbf{f}$. And the residual equation is: find $(\bar{\mathbf{u}},\bar{\theta}) \in X$ satisfying

$$-\Delta\bar{\mathbf{u}} + \nabla\bar{p} = \mathbf{f}(\mathbf{u}_N + \bar{\mathbf{u}}, \theta_N + \bar{\theta}) + \Delta\mathbf{u}_N - \nabla p_N, \tag{7a}$$
$$\nabla\cdot\bar{\mathbf{u}} = 0, \tag{7b}$$
$$-\Delta\bar{\theta} = g(\mathbf{u}_N + \bar{\mathbf{u}}, \theta_N + \bar{\theta}) + \Delta\theta_N. \tag{7c}$$

Set $\bar{F}(\bar{\mathbf{u}},\bar{\theta}) \equiv (\mathbf{f}(\mathbf{u}_N + \bar{\mathbf{u}}, \theta_N + \bar{\theta}) + \Delta\mathbf{u}_N - \nabla p_N, g(\mathbf{u}_N + \bar{\mathbf{u}}, \theta_N + \bar{\theta}) + \Delta\theta_N) \equiv (\bar{\mathbf{f}}(\bar{\mathbf{u}},\bar{\theta}), \bar{g}(\bar{\mathbf{u}},\bar{\theta}))$, then the Stokes operator $\mathcal{S}$ gives us a fixed point problem which is equivalent to (7):

$$(\bar{\mathbf{u}},\bar{\theta}) = \mathcal{S}\bar{F}(\bar{\mathbf{u}},\bar{\theta}) \equiv \mathcal{K}(\bar{\mathbf{u}},\bar{\theta}). \tag{8}$$

Since $X \subset H^1(\Omega)^4$, $\mathcal{K}$ is a compact operator in X . Hence by Schauder's fixed point theorem, if we find a nonempty, closed, convex, and bounded set $U \subset X$ satisfying $\mathcal{K}U \subset U$, then there exists a solution of (8) in U which is called a candidate set.

Define $\mathbf{P}_N : X \to X_N$ by $\mathbf{P}_N = (P_N, Q_N)$, then (2) can be simplified as: for $(\mathbf{u},\theta) \in X$

$$\langle \nabla((\mathbf{u},\theta) - \mathbf{P}_N(\mathbf{u},\theta)), \nabla(\mathbf{v},\vartheta)\rangle = 0, \quad \forall(\mathbf{v},\vartheta) \in X_N. \tag{9}$$

And (8) can be decomposed into two parts:

$$\mathbf{P}_N(\bar{\mathbf{u}},\bar{\theta}) = \mathbf{P}_N\mathcal{K}(\bar{\mathbf{u}},\bar{\theta}), \tag{10a}$$
$$(I - \mathbf{P}_N)(\bar{\mathbf{u}},\bar{\theta}) = (I - \mathbf{P}_N)\mathcal{K}(\bar{\mathbf{u}},\bar{\theta}). \tag{10b}$$

Define a Newton–like iteration operator $\mathcal{N} : X \to X_N$ for (8) and a new map $\mathcal{T}$ as follows:

$$\mathcal{N} \equiv \mathbf{P}_N - \mathcal{L}_N^{-1}\mathbf{P}_N(I - \mathcal{K}), \quad \mathcal{T} \equiv \mathcal{N} + (I - \mathbf{P}_N)\mathcal{K},$$

where $\mathcal{L}_N \equiv \mathbf{P}_N[I - \mathcal{S}F'(\mathbf{u}_N,\theta_N)]\big|_{X_N} : X_N \to X_N$ is assumed to be regular, i.e., one-to-one and onto. Here F' is the Fréchet derivative of F: for any $(\bar{\mathbf{u}},\bar{\theta}) \in X$,

$$F'(\mathbf{u},\theta)(\bar{\mathbf{u}},\bar{\theta}) \equiv \left(\mathbf{f}'(\mathbf{u},\theta)(\bar{\mathbf{u}},\bar{\theta}), g'(\mathbf{u},\theta)(\bar{\mathbf{u}},\bar{\theta})\right),$$
$$\mathbf{f}'(\mathbf{u},\theta)(\bar{\mathbf{u}},\bar{\theta}) \equiv -\frac{1}{\mathcal{P}}[(\mathbf{u}\cdot\nabla)\bar{\mathbf{u}} + (\bar{\mathbf{u}}\cdot\nabla)\mathbf{u}] + \mathcal{R}\bar{\theta}\mathbf{e}_z,$$
$$g'(\mathbf{u},\theta)(\bar{\mathbf{u}},\bar{\theta}) \equiv -\left[(\mathbf{u}\cdot\nabla)\bar{\theta} + (\bar{\mathbf{u}}\cdot\nabla)\theta\right] + \bar{w}.$$

The second part of $\mathcal{T}$ becomes small or a contraction if the truncation number N is sufficiently large. The operator $\mathcal{N}$ is compact since it maps X into the finite dimensional space X_N, and so is $\mathcal{T}$.

Lemma 4 *The problem (10) is equivalent to the following fixed point problem:*

$$(\bar{\mathbf{u}},\bar{\theta}) = \mathcal{T}(\bar{\mathbf{u}},\bar{\theta}). \tag{11}$$

From Lemma 4, we have a compatible verification condition of the form: $\mathcal{T}U \subset U$ if there exists a candidate set U which is nonempty, closed, convex, and bounded in X.

For given $\bar{\xi}_\alpha, \bar{\eta}_\alpha, \bar{\theta}_\alpha \ge 0$, set $[\bar{\xi}_\alpha] \equiv [-\bar{\xi}_\alpha, \bar{\xi}_\alpha]$, $[\bar{\eta}_\alpha] \equiv [-\bar{\eta}_\alpha, \bar{\eta}_\alpha]$, $[\bar{\theta}_\alpha] \equiv [-\bar{\theta}_\alpha, \bar{\theta}_\alpha]$, and define $U_N \subset X_N$ by

$$(\mathbf{u},\theta) \in U_N \iff \xi_\alpha \in [\bar{\xi}_\alpha], \eta_\alpha \in [\bar{\eta}_\alpha], \theta_\alpha \in [\bar{\theta}_\alpha]. \tag{12}$$

Let $X_N^\perp$ be the orthogonal complement of X_N in X. Given $m_1, m_2 \geq 0$, define $U_* \subset X_N^\perp$ by

$$(\mathbf{u}, \theta) \in U_*$$
$$\Longleftrightarrow \begin{cases} \|\mathbf{u}\|_0 \leq \frac{C_0^2}{(N+1)^2} m_1, & \|\nabla \mathbf{u}\|_0 \leq \frac{C_0}{N+1} m_1, \\ \|\mathbf{u}\|_\infty \leq \frac{2C_1}{\sqrt{N}} m_1, & \|\theta\|_0 \leq \frac{C_0^2}{(N+1)^2} m_2, \\ \|\nabla\theta\|_0 \leq \frac{C_0}{N+1} m_2. \end{cases} \quad (13)$$

Now, set $U \equiv U_N \oplus U_*$, then we obtain:

Theorem 5 *Let U_N, U_* and U be sets defined as above. If*

$$\mathcal{N}U \subset U_N, \quad (14a)$$
$$(I - \mathbf{P}_N)\mathcal{K}U \subset U_*. \quad (14b)$$

then there exists a fixed point of $\mathcal{T}$ in U.

In [4], proofs of Lemma 4 and Theorem 5 can be found with statements on computational verification conditions.

4. Numerical results

Solutions with 8 peaks are invariant under quarter-period diagonal translation in (x, y)-plane: $g(x+\pi/(2a), y+\pi/(2b), z) = g(x, y, z)$, which gives us more restriction, i.e, $\alpha_1 + \alpha_2$ is a multiple of 4 and α_1, α_2 are even. The size of finite part is reduced to one quarter of that of 2 peaks case. Thus we can use more large truncation numbers.

Finally, solutions with 32 peaks are invariant under $\frac{1}{8}$-period diagonal translation in (x, y)-plane, i.e., $g(x + \pi/(4a), y+\pi/(4b), z) = g(x, y, z)$ which means that $\alpha_1+\alpha_2$ is a multiple of 8 and α_1, α_2 are multiples of 4. The reduction is one sixteenth of 2 peaks case and one quarter of 8 peaks.

We set $a^2 = \frac{1}{8}$, $b^2 = \frac{3}{8}$ and $\mathcal{P} = 10$ in the numerical experiments with 1% inflation factor ($\delta = 0.01$). Then the critical Rayleigh number $\mathcal{R}_c = 6.75$ can be attained at some special mode α, for example, $\alpha = (2, 0, 1)$ or $\alpha = (1, 1, 1)$ (see [6] for detail).

For the interval arithmetic, we use PROFIL [2] and INTLAB [1] on Intel Pentium 4 (3.8 GHz) machine.

Tables show that the relative Rayleigh number $r = \mathcal{R}/\mathcal{R}_c$, the truncation number N, the converged step k, L^∞ norms of finite parts $(\mathbf{u}_h, \theta_h)$, and the bounds m_1, m_2 of infinite part. Here, L^1 type estimate is used to calculate upper bounds of L^∞ norms.

In figures, isothermal lines are drawn after adding the conduction solution [3], and streamlines are also supplied on the slice of the middle of the domain ($z = \frac{\pi}{2}$).

The bifurcation diagram is drawn for L^∞ norms of approximate solutions $(\mathbf{u}_N, \theta_N)$ with 2, 8 and 32 peaks along the relative Rayleigh number.

r	N	k	$\|\nabla(\mathbf{u}_h, \theta_h)\|_\infty$	m_1	m_2
2.02	22	6	2×10^{-9}	4×10^{-10}	2×10^{-9}
2.1	22	8	2×10^{-6}	9×10^{-8}	3×10^{-7}
2.2	22	11	3×10^{-5}	2×10^{-6}	4×10^{-6}
2.4	32	13	6×10^{-6}	3×10^{-9}	2×10^{-8}

Table 1: Verification results for rectangular type solutions with 8 peaks.

r	N	k	$\|\nabla(\mathbf{u}_h, \theta_h)\|_\infty$	m_1	m_2
2.02	22	7	2×10^{-8}	3×10^{-9}	8×10^{-9}
2.1	22	10	1×10^{-5}	6×10^{-7}	2×10^{-6}
2.2	28	13	7×10^{-6}	5×10^{-8}	3×10^{-7}
2.4	32	44	5×10^{-5}	2×10^{-7}	8×10^{-7}

Table 2: Verification results for hexagonal type solutions with 8 peaks.

r	N	k	$\|\nabla(\mathbf{u}_h, \theta_h)\|_\infty$	m_1	m_2
13.635	40	7	6×10^{-7}	9×10^{-10}	2×10^{-9}
14.0	40	8	2×10^{-5}	5×10^{-8}	7×10^{-8}
14.175	40	9	3×10^{-5}	2×10^{-7}	2×10^{-7}
14.85	48	10	6×10^{-7}	2×10^{-8}	3×10^{-8}
16.2	62	12	4×10^{-8}	5×10^{-10}	2×10^{-9}

Table 3: Verification results for rectangular type solutions with 32 peaks.

r	N	k	$\|\nabla(\mathbf{u}_h, \theta_h)\|_\infty$	m_1	m_2
13.635	40	7	3×10^{-6}	5×10^{-9}	6×10^{-9}
14.0	40	9	8×10^{-5}	3×10^{-7}	4×10^{-7}
14.175	40	11	2×10^{-4}	1×10^{-6}	3×10^{-6}
14.85	48	13	6×10^{-4}	2×10^{-7}	3×10^{-7}
16.2	62	47	2×10^{-6}	2×10^{-8}	6×10^{-8}

Table 4: Verification results for hexagonal type solutions with 32 peaks.

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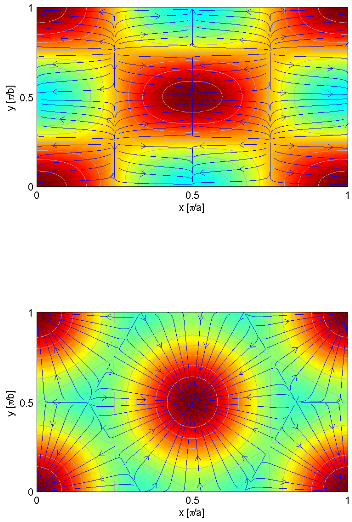

Figure 1: Isothermal lines and streamlines of rectangular and hexagonal type solutions with 8 peaks on the plane $[0, \frac{\pi}{a}] \times [0, \frac{\pi}{b}] \times \left\{\frac{\pi}{2}\right\}$ when $r = \mathcal{R}/\mathcal{R}_c = 2.4$.

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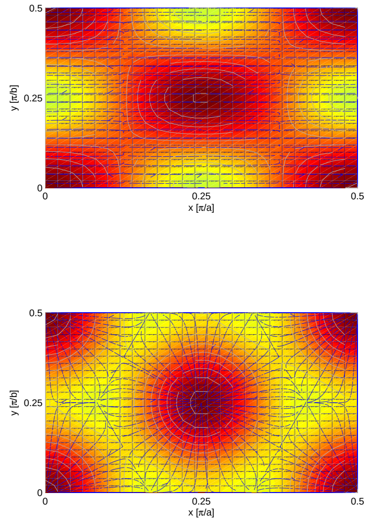

Figure 2: Isothermal lines and streamlines of rectangular and hexagonal type solutions with 32 peaks on the plane $[0, \frac{\pi}{2a}] \times [0, \frac{\pi}{2b}] \times \left\{\frac{\pi}{2}\right\}$ when $r = \mathcal{R}/\mathcal{R}_c = 16.2$.

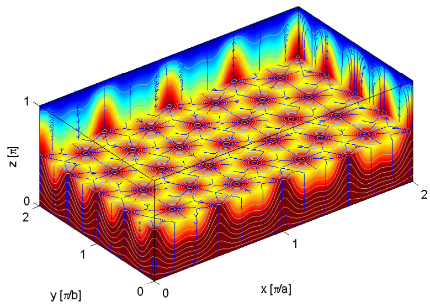

Figure 3: Isothermal lines and streamlines of the hexagonal type solution with 32 peaks when $r = \mathcal{R}/\mathcal{R}_c = 16.2$.

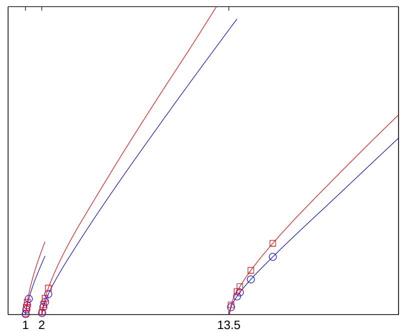

Figure 4: Bifurcation diagrams for 2, 8 and 32 peaks with respect to the relative Rayleigh number. Red for hexagonal, blue for rectangular cases. Red rectangles and blue circles are verified points.