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Fast Computation of Approximated Error Bound for Harmonic Balance Method Using Algebraic Representation

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Abstract—Harmonic balance (HB) method is well known for analyzing periodic oscillations on nonlinear circuit systems. Because the HB method is an approximation method, approximated solutions have been guaranteed by an error bound. However, its computation is very time-consuming compared with solving the HB equation. This paper proposes a fast computational method of an approximated error bound using an algebraic representation based on Gröbner base. The proposed method decrease the computational cost of the error bound considerably. Using the proposed method, we can guarantee the solutions as fast as solving the HB equation.

1. Introduction

Harmonic balance (HB) method is well known for analyzing periodic oscillations on nonlinear circuit systems [1,2]. A circuit equation is transposed to simultaneous algebraic equation called HB equation due to an approximation by truncated Fourier series of variables. Because the HB method, which ignores high frequency components, is the approximation method, approximated solutions of the HB equation have been guaranteed by an error bound [3–5]. However, the computation of the error bound is very time-consuming compared with solving the HB equation.

This paper proposes a fast computational method of an approximated error bound for the HB method using an algebraic representation of the error bound. The report [8] presents a method to obtain the algebraic representation using Gröbner base [6,7]. However, the computational cost of the Gröbner base is highly dependent on the complexity of the equations of the error bound. Especially, the computational cost increases exponentially according to the expansion of the number of considered frequency components. Thus, the number of the considered frequency components makes the method in [8] difficult.

In order to overcome the difficulty, we propose an efficient method to obtain the algebraic representation of the error bound using transformations of variables. The proposed method is not dependent on the number of the considered frequency components. Using the algebraic representation of the error bound, we can approximate the error bound to the quadratic form. The quadratic approximation decreases the computational cost of the error bound considerably. Using the proposed method, we can guarantee the solutions as fast as solving the HB equation.

Section 2 describes the HB method, and Section 3 reviews the method to obtain the error bound for the HB method. In Section 4, we propose the efficient method to obtain the algebraic representation of the error bound and the quadratic approximation of the error bound using the algebraic representation. In Section 5, we apply the proposed method to Duffing equation as example. Finally, conclusions are drawn in Section 6.

2. Harmonic Balance (HB) Method

We consider a nonlinear feedback system shown in Figure 1. The circuit equation of the system is written by

$$u(\tau) = G(s)\,\{e(\tau) - N[u(\tau)]\}\,, \tag{1}$$

$$N[u(\tau)] = \sum_{i=0}^{p} c_{2i+1} u^{2i+1}, \tag{2}$$

where $s = \mathrm{d}/\mathrm{d}\tau$, $N[u]$ is a polynomial type nonlinear element and a monotone increasing function of u, and transfer function $G(s)$ has a low-pass characteristics.

Assuming that Eq.(1) has a periodic solution, we write

$$u(\tau) \equiv \sum_{k=0}^{\infty} \Re\left[\dot{x}_k e^{jk\tau}\right] = \sum_{k=0}^{\infty} \Re\left[(x_{k\mathrm{r}} + \mathrm{j}x_{k\mathrm{s}})e^{\mathrm{j}k\tau}\right], \tag{3}$$

where $\dot{x}_k(\tau) \in \mathbb{C}$, $\mathbb{C}$ is a set of complex numbers, $x_{0\mathrm{s}} = 0$ and $\Re[\cdot]$ denotes the real part of $\cdot$. We assume that the above solution can be approximated by truncated Fourier series with n frequency components

$$u_{\mathrm{L}}(\tau) = K_{\mathrm{L}} u(\tau) = \sum_{k=0}^{n} \Re\left[(x_{k\mathrm{r}} + \mathrm{j}x_{k\mathrm{s}})e^{\mathrm{j}k\tau}\right], \tag{4}$$

where K_{L} is a projection operator that expresses the truncation of the Fourier series. The substitution of Eq.(4) into Eq.(1) gives

$$\sum_{k=0}^{n} \Re\left[\{\dot{x}_k - G(\mathrm{j}k)\,(\dot{e}_k + \dot{y}_k)\}\, e^{\mathrm{j}k\tau}\right] = 0, \tag{5}$$

where

$$\dot{y}_k = \frac{1}{\pi}\int_0^{2\pi} N[u(\tau)]e^{-\mathrm{j}k\tau}\mathrm{d}\tau, \qquad \dot{y}_0 = \frac{1}{2\pi}\int_0^{2\pi} N[u(\tau)]\mathrm{d}\tau,$$

$$\dot{e}_k = \frac{1}{\pi}\int_0^{2\pi} e(\tau)\, e^{-\mathrm{j}k\tau}\mathrm{d}\tau, \qquad \dot{e}_0 = \frac{1}{2\pi}\int_0^{2\pi} e(\tau)\mathrm{d}\tau,$$

and $s^n e^{\mathrm{j}\tau} = (\mathrm{j}k)^n e^{\mathrm{j}\tau}$ is satisfied. By this relation, HB equation is written by

$$\boldsymbol{f}(\boldsymbol{x}) \equiv (f_{0\mathrm{r}}, f_{1\mathrm{r}}, f_{1\mathrm{s}}, \ldots, f_{n\mathrm{r}}, f_{n\mathrm{s}})^{\mathrm{T}} = \boldsymbol{0} \in \mathbb{R}^{2n+1}, \tag{6}$$

$$f_{k\mathrm{r}} \equiv \Re\left[\dot{x}_k - G(\mathrm{j}k)\,\{\dot{e}_k + \dot{y}_k\}\right],\; f_{0\mathrm{r}} = \Re\left[\dot{x}_0 - G(0)\,\{\dot{e}_0 + \dot{y}_0\}\right],$$
$$f_{k\mathrm{s}} \equiv \Im\left[\dot{x}_k - G(\mathrm{j}k)\,\{\dot{e}_k + \dot{y}_k\}\right],\; f_{0\mathrm{s}} = 0,$$
$$\boldsymbol{x} \equiv (x_{0\mathrm{r}}, x_{1\mathrm{r}}, x_{1\mathrm{s}}, \ldots, x_{n\mathrm{r}}, x_{n\mathrm{s}})^{\mathrm{T}},\; k = 0, \ldots, n,$$

where $\mathbb{R}$ is a set of real numbers and $\Im[\cdot]$ denotes the imaginary part of $\cdot$.

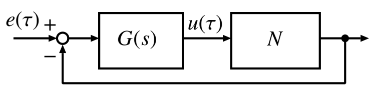

Figure 1: Nonlinear feedback system.

3. Error Bound of HB Method

3.1. Estimation of High Frequency Components

We define a projection operator $K_{\rm H}$;

$$u_{\rm H}(\tau) = K_{\rm H}u(\tau) = \sum_{k=n+1}^{\infty} \Re\left[(x_{k{\rm r}} + {\rm j}x_{k{\rm s}})e^{{\rm j}k\tau}\right], \quad (7)$$

$$u(\tau) = u_{\rm L}(\tau) + u_{\rm H}(\tau), \ \ K_{\rm L} + K_{\rm H} = I,$$

where I is an identity operator. Applying the operator $K_{\rm H}$ to Eq.(1), we obtain

$$u_{\rm H}(\tau) = -K_{\rm H}G(s)N[u_{\rm L}(\tau) + u_{\rm H}(\tau)]. \quad (8)$$

We define λ as

$$\left\| \frac{{\rm d}N[u(\tau)]}{{\rm d}u} \right\|_\infty \leq \lambda, \quad \forall u \in \{u \mid \|u\|_1 \leq c\}, \quad (9)$$

where c denotes a certain value and norms are defined by

$$l^{\,i}\text{norm}: \ \|u(\tau)\|_i \equiv \left\{ \textstyle\sum_{k=0}^{\infty} \left(\sqrt{x_{k{\rm r}}^2 + x_{k{\rm s}}^2} \right)^i \right\}^{1/i}, \ i = 1, 2, \quad (10)$$

$$L^\infty\text{norm}: \quad \|u(\tau)\|_\infty \equiv \sup_{\tau\in[0,2\pi)} |u(\tau)|. \quad (11)$$

Using mean value theorem and contraction mapping theorem, we can obtain the estimation of the high frequency components for l^1 and l^2 norms;

$$\|u_{\rm H}\|_i \leq \frac{\|F\|_\infty}{1 - \|F\|_\infty}\|u_{\rm L}\|_i, \ i = 1, 2, \quad (12)$$

where $\|F\|_\infty \equiv \lambda \sup_{|k|>n} \left|G({\rm j}k)\right| < 1$. Using Eq.(12), we write λ as

$$\lambda = \sum_{i=0}^{p}(2i + 1)c_{2i+1}\left(1 + \frac{\|F\|_\infty}{1 - \|F\|_\infty}\right)^{2i} \|u_{\rm L}\|_1{}^{2i}. \quad (13)$$

Then, Eq.(13) is rewritten by

$$f_{\rm eb1}(\lambda, \boldsymbol{x}) = \lambda(1 - \lambda H)^{2p} - \sum_{i=0}^{p}(2i+1)c_{2i+1}(1-\lambda H)^{2(p-i)}\left(\sum_{k=0}^{n}\sqrt{x_{k{\rm r}}^2 + x_{k{\rm s}}^2}\right)^{2i} = 0, \quad (14)$$

where $H = \sup_{|k|>n} \left|G({\rm j}k)\right| \in \mathbb{R}$ is constant.

3.2. Error Bound by Homotopy Invariance

In order to guarantee the solutions of the HB equation, we consider the equation which provides the HB equation;

$$\text{FH}(u_{\rm L}) \equiv G(s)^{-1}u_{\rm L}(\tau) - \{e(\tau) + K_{\rm L}N[u_{\rm L}(\tau)]\} = 0. \quad (15)$$

Using homotopy invariance theorem and Eq.(15), we obtain the equation of the error bound for the HB method;

$$\|\text{FH}(u_{\rm L})\|_2 = \frac{\lambda\|F\|_\infty}{1 - \|F\|_\infty}\|u_{\rm L}\|_2. \quad (16)$$

We rewrite the Eq.(16) as follows;

$$f_{\rm eb2}(\lambda, \boldsymbol{x}) = \lambda^4 H^2 \sum_{k=0}^{n}\left(x_{k{\rm r}}^2 + x_{k{\rm s}}^2\right) - (1-\lambda H)^2 \sum_{k=0}^{n}\left(f_{k{\rm r}}^2 + f_{k{\rm s}}^2\right) = 0. \quad (17)$$

The equation (17) is an algebraic equation.

The equations obtained by Eq.(14) and Eq.(17) are the algebraic equations with λ and $\boldsymbol{x}$. Using the elimination of λ by the Gröbner base of lexicographic order $\lambda \succ x_{k{\rm r}}$ (or $\lambda \succ x_{k{\rm s}}$) from the equations [6, 7], we can obtain the algebraic representation $g_{\rm eb}(\boldsymbol{x})$ of the error bound for the HB method in $2n + 1$ dimensional space [8]. However, Eq.(14) and Eq.(17) become complicated according to the expansion of the number of considered frequency components. Thus, the algebraic representation $g_{\rm eb}(\boldsymbol{x})$ can not be calculated by the method in [8] when we consider more than 2 frequency components.

4. Algebraic Representation of Error Bound

4.1. Efficient Method Using Transformations of Variables

In order to overcome the difficulty of the method in [8], we propose an efficient method to obtain the algebraic representation of the error bound by transformations of variables. Because the number n of the considered frequency components complicates only the norms in Eq.(14) and Eq.(17), we transform the norms into new variables;

$$\alpha(\boldsymbol{x}) \equiv \|u_{\rm L}\|_1 = \sum_{k=0}^{n}\sqrt{x_{k{\rm r}}^2 + x_{k{\rm s}}^2}, \ \ \beta(\boldsymbol{x}) \equiv \|u_{\rm L}\|_2{}^2 = \sum_{k=0}^{n}\left(x_{k{\rm r}}^2 + x_{k{\rm s}}^2\right),$$

$$\gamma(\boldsymbol{x}) \equiv \|\text{FH}(u_{\rm L})\|_2{}^2 = \sum_{k=0}^{n}\left(f_{k{\rm r}}^2(\boldsymbol{x}) + f_{k{\rm s}}^2(\boldsymbol{x})\right). \quad (18)$$

Thus, Eq.(14) and Eq.(17) are represented by

$$f_{\rm eb1}(\lambda, \alpha) = \lambda(1-\lambda H)^{2p} - \sum_{i=0}^{p}(2i+1)c_{2i+1}(1-\lambda H)^{2(p-i)}\alpha^{2i} = 0, \quad (19)$$

$$f_{\rm eb2}(\lambda, \beta, \gamma) = \lambda^4 H^2 \beta - (1-\lambda H)^2\gamma = 0. \quad (20)$$

Eq.(19) and Eq.(20) do not depend on n because only the new variables α, β, γ change if n changes.

Then, $g_{\rm eb}(\alpha,\beta,\gamma)$ is obtained by the elimination of λ using the Gröbner base. Since Eq.(19) and Eq.(20) are simpler than Eq.(14) and Eq.(17), the computational cost of $g_{\rm eb}(\alpha,\beta,\gamma)$ is less than the cost of $g_{\rm eb}(\boldsymbol{x})$ in [8]. Finally, the algebraic representation of the error bound $g_{\rm eb}(\boldsymbol{x})$ is obtained by the substitution of α, β, γ into the $g_{\rm eb}(\alpha,\beta,\gamma)$. Thus, the algorithm is written by

- S1. We give the algebraic equations $f_{\rm eb1}(\lambda, \alpha) = 0$ and $f_{\rm eb2}(\lambda, \beta, \gamma) = 0$.
- S2. We obtain $g_{\rm eb}(\alpha,\beta,\gamma)$ by the elimination of λ using on the Gröbner base of order $\lambda \succ (\alpha,\beta,\gamma)$ from $f_{\rm eb1}$ and $f_{\rm eb2}$.
- S3. We determine the variables α, β and γ based on the number n of the considered frequency components.
- S4. We obtain the algebraic representation of the error bound $g_{\rm eb}(\boldsymbol{x})$ by the substitution of Eq.(18) into the $g_{\rm eb}(\alpha,\beta,\gamma)$.

The algebraic representation $g_{\rm eb}(\alpha,\beta,\gamma)$ is not dependent on n due to the independence of Eq.(19) and Eq.(20) on n. That is, we can obtain the algebraic representation of the error bound even if we consider many frequency components.

4.2. Quadratic Approximation of Error Bound Using Algebraic Representation

In order to estimate the error bound, we must project the error bound to the complex plane of a target frequency component. The report [5] evaluates the projection of the error bound by the sufficient condition. However, the computation of the projection is very time-consuming compared with solving the HB equation. For example, the method in [5] requires to solve the simultaneous equations many times.

This paper presents a fast computational method to obtain the projection of the error bound using the algebraic representation. The error bound is a neighborhood of the solution of the HB equation, and resembles an ellipsoidal body, i.e., a quadratic form. Thus, we can approximate the error bound to the quadratic form using variations of the solution of the HB equation. The quadratic approximation of the error bound enables to express its projection as the quadratic form.

Let us consider that the error bound is projected to a complex plane (x_1, x_2) where $x_1, x_2 \in \{x_{0\mathrm{r}}, x_{1\mathrm{r}}, x_{1\mathrm{s}}, \ldots, x_{n\mathrm{r}}, x_{n\mathrm{s}}\}$ and we rewrite the variables $x_{0\mathrm{r}}, x_{1\mathrm{r}}, x_{1\mathrm{s}}, \ldots, x_{n\mathrm{r}}, x_{n\mathrm{s}}$ except for x_1, x_2 to $x_3, x_4, \ldots, x_N$ where $N = 2n + 1$. Using the variations $\Delta x_1, \Delta x_2, \ldots, \Delta x_N$ of the solution of the HB equation and Taylor expansion, we obtain the quadratic approximation $g_{\mathrm{eb}}(\boldsymbol{x})$ of the error bound as follows;

$$g_{\mathrm{eb}}(\boldsymbol{x}) \approx g_{\Delta\mathrm{eb}}(\Delta x_1, \Delta x_2, \ldots, \Delta x_N) = \sum_{i=1}^{N} a_{ii}\Delta x_i^2 + 2\sum_{i=1}^{N}\sum_{j=i+1}^{N} a_{ij}\Delta x_i \Delta x_j + 2\sum_{i=1}^{N} a_{0i}\Delta x_i + a_{00}. \quad (21)$$

Then, the quadratic approximation is written by

$$g_{\Delta\mathrm{eb}}(\Delta\boldsymbol{x}) = \Delta\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\Delta\boldsymbol{x} = 0, \quad (22)$$

where

$$\boldsymbol{A} \equiv \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0N} \\ a_{01} & a_{11} & \cdots & a_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{0N} & a_{1N} & \cdots & a_{NN} \end{bmatrix} \in \mathbb{R}^{(N+1)\times(N+1)}, \quad \Delta\boldsymbol{x} \equiv \begin{bmatrix} 1 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}.$$

In order to obtain the projection of the approximated error bound $g_{\Delta\mathrm{eb}}(\Delta\boldsymbol{x})$, we decompose $\boldsymbol{A}$ and $\Delta\boldsymbol{x}$ into

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{A}_2^{\mathrm{T}} & \boldsymbol{A}_3 \end{bmatrix}, \ \Delta\boldsymbol{x} = \begin{bmatrix} \Delta\boldsymbol{x}_1 \\ \Delta\boldsymbol{x}_2 \end{bmatrix}, \ \Delta\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix}, \ \Delta\boldsymbol{x}_2 = \begin{bmatrix} x_3 \\ \vdots \\ x_N \end{bmatrix} \quad (23)$$

where partial matrices of $\boldsymbol{A}$ denote $\boldsymbol{A}_1 \in \mathbb{R}^{3\times3}$, $\boldsymbol{A}_2 \in \mathbb{R}^{(N-2)\times3}$ and $\boldsymbol{A}_3 \in \mathbb{R}^{(N-2)\times(N-2)}$, respectively. Because the projection of $g_{\Delta\mathrm{eb}}(\Delta\boldsymbol{x})$ satisfies

$$\frac{\partial g_{\Delta\mathrm{eb}}(\Delta\boldsymbol{x})}{\partial \Delta x_k} = 0, \quad k = 3, \ldots, N, \quad (24)$$

we obtain a relation;

$$\boldsymbol{A}_2^{\mathrm{T}}\Delta\boldsymbol{x}_1 + \boldsymbol{A}_3\Delta\boldsymbol{x}_2 = \boldsymbol{0}. \quad (25)$$

Thus, the projection of the approximated error bound is represented by

$$\begin{aligned} \Delta\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\Delta\boldsymbol{x} &= \left(\Delta\boldsymbol{x}_1^{\mathrm{T}}, \Delta\boldsymbol{x}_2^{\mathrm{T}}\right)^{\mathrm{T}} \begin{bmatrix} \boldsymbol{A}_1\Delta\boldsymbol{x}_1 + \boldsymbol{A}_2\Delta\boldsymbol{x}_2 \\ \boldsymbol{0} \end{bmatrix} \\ &= \Delta\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{A}_1\Delta\boldsymbol{x}_1 + \Delta\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{A}_2\Delta\boldsymbol{x}_2 \\ &= \Delta\boldsymbol{x}_1^{\mathrm{T}}\left(\boldsymbol{A}_1 - \boldsymbol{A}_2\boldsymbol{A}_3^{-1}\boldsymbol{A}_2^{\mathrm{T}}\right)\Delta\boldsymbol{x}_1 = 0. \quad (26) \end{aligned}$$

The quadratic approximation algorithm is written by

S1. We calculate the algebraic representation $g_{\mathrm{eb}}(\boldsymbol{x})$ of the error bound using the Gröbner base.

S2. We set the target complex plane (x_1, x_2) and other variables $x_3, \ldots, x_N$.

S3. We obtain algebraic representations of the elements a_{ij}, $(i, j = 0, \ldots, N, i \leq j)$ of the matrix $\boldsymbol{A}$ with the solution of the HB equation and the circuit parameters.

S4. We determine a_{ij} by the substitution of the given solution and parameters into the algebraic representations a_{ij}, $(i, j = 0, \ldots, N, i \leq j)$.

S5. We obtain the projection of the approximated error bound by $\boldsymbol{A}_1 - \boldsymbol{A}_2\boldsymbol{A}_3^{-1}\boldsymbol{A}_2^{\mathrm{T}}$.

Using the approximated error bound, the projection of the error bound can be plotted easily because the number of the variables in Eq.(26) is only 2 and the maximum degree of Eq.(26) is 2.

5. Example

5.1. Duffing Equation

We apply the proposed method to Duffing equation;

$$\frac{\mathrm{d}^2 u(\tau)}{\mathrm{d}\tau^2} + \mu\frac{\mathrm{d}u(\tau)}{\mathrm{d}\tau} + u^3 = E\cos\tau, \quad (27)$$

where, the Duffing equation can be described as Eq.(1);

$$G(s) = \frac{1}{s^2 + \mu s}, \quad N[u(\tau)] = u^3. \quad (28)$$

We apply the HB method to Eq.(27). We assume that the direct current and even harmonics equal zero $x_{0\mathrm{r}} = 0$, $x_{k\mathrm{r}} = x_{k\mathrm{s}} = 0, (1 \leq k \leq n, k = 0 \bmod 2)$ for simplicity.

5.2. Algebraic Representation of Error Bound

The equations f_{eb1} and f_{eb2} is written by

$$f_{\mathrm{eb1}}(\lambda, \alpha) = \lambda(1 - \lambda H)^2 - 3\alpha^2 = 0, \quad (29)$$

$$f_{\mathrm{eb2}}(\lambda, \beta, \gamma) = \lambda^4 H^2 \beta - (1 - \lambda H)^2 \gamma = 0. \quad (30)$$

Thus, the following algebraic representation $g_{\mathrm{eb}}(\alpha, \beta, \gamma)$ of the error bound is obtained by the elimination of λ using the Gröbner base of order $\lambda \succ (\alpha, \beta, \gamma)$;

$$\begin{aligned} g_{\mathrm{eb}}(\alpha, \beta, \gamma) = {} & 9\alpha^4 H^6\gamma^3 - 135\alpha^4\beta H^4\gamma^2 \\ & -6\alpha^2\beta H^3\gamma^2 - 270\alpha^6\beta^2 H^3\gamma + 225\alpha^4\beta^2 H^2\gamma \\ & -30\alpha^2\beta^2 H\gamma + \beta^2\gamma - 81\alpha^8\beta^3 H^2 = 0. \quad (31) \end{aligned}$$

In order to compare the computational cost between the proposed method and the method in [8], the computational cost of the error bound using the Gröbner base of $\lambda \succ \alpha \succ \beta \succ \gamma$ and $\lambda \succ x_{1\mathrm{r}} \succ x_{1\mathrm{s}}$ is shown in Table 1 where $n = 1$. Thus, we can confirm an efficiency of the proposed method.

Further, Eq.(31) is not dependent on the number of the considered frequency components. Thus, we can obtain the algebraic representation $g_{\mathrm{eb}}(\boldsymbol{x})$ even if we consider many frequency components.

5.3. Quadratic Approximation of Error Bound

We apply the quadratic approximation of the error bound to this example. Let us consider $x_1 = x_{1\mathrm{r}}, x_2 = x_{1\mathrm{s}}, x_3 = x_{3\mathrm{r}}, \ldots, x_{N-1} = x_{n\mathrm{r}}, x_N = x_{n\mathrm{s}}$, and $f_1 = f_{1\mathrm{r}}, f_2 = f_{1\mathrm{s}}, f_3 = f_{3\mathrm{r}}, \ldots, f_{N-1} = f_{n\mathrm{r}}, f_N = f_{n\mathrm{s}}$ where $N = 2n$. The matrix $\boldsymbol{A}$ of the approximated error bound is represented by

$$a_{00} = -81H^2\alpha_0^8\beta_0, \quad (32)$$

$$a_{0i} = -81H^2\alpha_0^8 x_i + 4\alpha_0^7 a_{0i}\beta_0, \quad (33)$$

$$\begin{aligned} a_{ii} = {} & (225\alpha_0^4 H^2 - 30\alpha_0^4 H - 270\alpha_0^6 H^3 + 1)\sum_{k=1}^{N}\left(\frac{\partial f_k(\mu, E)}{\partial x_i}\right)^2 \\ & -81H^2\alpha_0^6(16\alpha_0\alpha_{0i}x_i + 8\alpha_0\alpha_{ii}\beta_0 + 28\alpha_{0i}^2\beta_0 + \alpha_0^2), \quad (34) \end{aligned}$$

$$\begin{aligned} a_{ij} = {} & (225\alpha_0^4 H^2 - 30\alpha_0^4 H - 270\alpha_0^6 H^3 + 1)\sum_{k=1}^{N}\left(\frac{\partial f_k(\mu, E)}{\partial x_j}\frac{\partial f_k(\mu, E)}{\partial x_i}\right) \\ & -81H^2\alpha_0^6(8\alpha_0\alpha_{0i}x_j + 8\alpha_0\alpha_{0j}x_i + 28\alpha_{0i}\alpha_{0j}\beta_0 + 4\alpha_0\alpha_{ij}\beta_0), \quad (35) \end{aligned}$$

Table 1: Comparison of computational cost between proposed method and method in [8] ($n = 1$).

Method	Order of Gröbner base	Computation time [s]	Required memory [MB]
Method in [8]	$\lambda \succ x_{1\mathrm{r}} \succ x_{1\mathrm{s}}$	7425	956
Proposed method	$\lambda \succ \alpha \succ \beta \succ \gamma$	0.007	1.09

Calculated by a PC with Xeon 3.06GHz CPU.

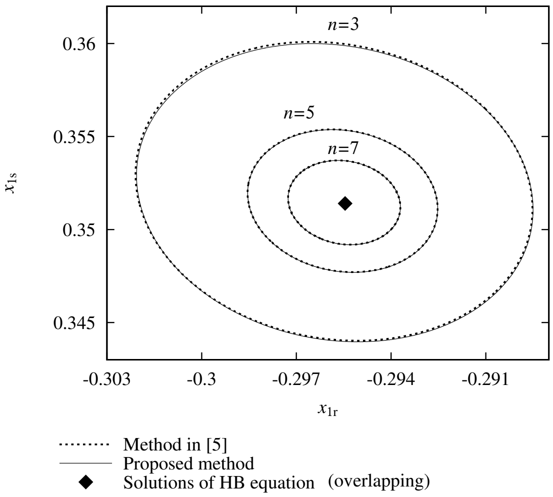

Figure 2: Projections of error bound for HB method on the (x_{1r}, x_{1s}) plane ($n = 3, 5, 7,\ \mu = 1$, and $E = 0.6$).

where

$$\alpha_0 = \sum_{k=1}^{n} \sqrt{x_{2k}^2 + x_{2k+1}^2},\ \ \alpha_{0i} = \frac{x_i}{\hat{\alpha}_i},\ \ \alpha_{ii} = \frac{x_i}{2\hat{\alpha}_i} - \frac{x_i^2}{2\hat{\alpha}_i{}^3},\ \ \beta_0 = \sum_{k=1}^{N} x_k^2,$$

$$\alpha_{ij} = \begin{cases} -\frac{x_i x_j}{2\hat{\alpha}_i} & |i-j| = 1 \\ 0 & |i-j| \neq 1 \end{cases},\ \ \hat{\alpha}_i = \begin{cases} \sqrt{x_i^2 + x_{i+1}^2} & i = 1 \bmod 2 \\ \sqrt{x_{i-1}^2 + x_i^2} & i = 0 \bmod 2 \end{cases},$$

$$H = \frac{1}{(n+1)\sqrt{(n+1)^2 + \mu^2}},\ \ i = 1, \ldots, N,\ j = 2, \ldots, N,\ i < j.$$

In order to confirm the validity of the approximation, the projections by the proposed method and the method in [5] are shown in Figure 2 where $\mu = 1,\ E = 0.6,\ n = 3, 5, 7$ and the solutions of the HB equation are overlapping. The projection of the approximated error bound is close to the projection in [5].

Moreover, the projection of the approximated error bound with the parameter E changing from 0.3 to 0.7 is shown in Figure 3 where $\mu = 1$ and $n = 3$. Because the elements (32), (33), (34) and (35) of the matrix $\boldsymbol{A}$ contain the circuit parameters symbolically, the approximated error bound can be easily obtained even if we change the circuit parameters such as Figure 3.

Further, the computational time of the proposed method and the method in [5] for $n = 3, 5, 7, 19$ is shown in Table 2 when we vary the parameters μ from 1 to 5, E from 0.3 to 0.7. Additionally, we also show the solving time of the HB equation in Table 2. Although the proposed method in Table 2 does not contain the computational cost of $g_{\mathrm{eb}}(\boldsymbol{x})$, $g_{\mathrm{eb}}(\boldsymbol{x})$ is calculated only once and the computational cost is very low such as Table 1. Thus, we can confirm that the proposed method reduces the computational cost of the error bound dramatically. Although the conventional method is very time-consuming compared with solving the HB equation, the proposed method guarantees the solutions as fast as solving the HB equation.

6. Conclusion

This paper proposed a fast computational method of an error bound for HB method using an algebraic representation of the error bound based on Gröbner base. In order to obtain the algebraic representation of the error bound, we presented an efficient method using transformations of variables. The proposed method does not depend on the number of considered frequency components. The algebraic representation enables to approximate the error bound to a quadratic form. Further, we confirmed that the quadratic approximation enables to guarantee the solutions as fast as solving the HB equations.

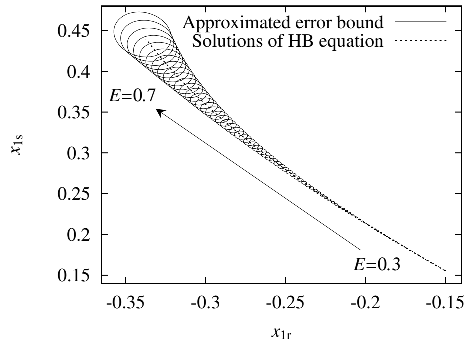

Figure 3: Projection of approximated error bound with parameter E changing from 0.3 to 0.7 ($n = 3$ and $\mu = 1$).

Table 2: Computational time of projection of error bound [s] (μ varied from 1 to 5 and E varied from 0.3 to 0.7, using Newton method with $40 \times 100 \times 32$ points).

Method	$n = 3$	$n = 5$	$n = 7$	$n = 19$
HB method	5.928	9.080	15.101	58.188
Method in [5]	461.761	685.111	1028.851	3032.523
Proposed method	20.694	22.652	23.607	46.960

Calculated by a PC with Xeon 3.06GHz CPU.

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