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The interplay between nonlinear analysis and numerical analysis for semilinear elliptic problems.

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Abstract—The idea of this talk is to explore the interplay between the theorems of nonlinear analysis of semilinear elliptic partial differential equations and the numerical schemes that are used to find their approximate solutions. In particular, we study how theorems of symmetry and a priori boundedness for the continuous case can shed light on what can be proved for discrete approximation.

1. Introduction

A classic paper is that of Gidas, Ni and Nirenberg [**?**], in which a typical result of the type we have in mind was: a positive solution of the boundary value problem

$$\Delta u = f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1}$$

must be radially symmetric if Ω is a ball.

A related area has been attracting growing attention, namely how does one approximate solutions of this type of nonlinear boundary value problem? Typically, the work in this area relies on a suitable discretization of (1), (most commonly by finite-differences), and then uses theoretical ideas from nonlinear analysis such as monotonicity methods, mountain pass algorithms, or linking methods, to develop an approximate or exact solution to the discretized problem.

Usually, the discretized problem is only approximately solved. The conventional wisdom is that if refinement of the grid results in only small changes to the approximate solution, then one is in the vicinity of a true solution of (1). In some cases, it is possible, using exact arithmetic, and eigenvalue estimates, to give a computer-assisted proof that there is a true solution of (1) in the vicinity of an approximate one. For a complete list of references to this problem, see [2].

One subject of this talk is to begin to address the so-far-neglected question: if the partial differential equation (1) has inherited certain symmetry properties from the domain, to what extent does the discretized problem also inherit these symmetry properties? After all, the discretized problem is supposed to be a close approximation of the continuous problem, at least for high-dimensional discretization. If the symmetries are not reflected in the solution, important properties of the solution are being missed.

An example from ordinary differential equations is illustrative. Here, the symmetry property of positive solutions of

$$u'' = f(u), \quad u > 0 \text{ in } (-L, L), \quad u(-L) = u(L) = 0. \tag{2}$$

are elementary.

Note that a positive solution u of (2) must have a maximum at some point $x_0 \in (-L, L)$. Then observe that since $u(x)$ and $u(2x_0 - x)$ both satisfy the same initial value problem at x_0, they must be identical. From this we can conclude that $x_0 = 0$ and that the solution is monotone decreasing on $(0, L)$. (This also follows from the paper of Gidas, Ni, and Nirenberg.)

The first natural conjecture would be that the discrete approximate solution u would have a maximum at $i = 0$, and be symmetric about 0 in the sense that $u_{-i} = u_i$. This would exactly reflect the symmetry properties of the analogous continuous problem. Unfortunately, this conjecture is easily shown to be false.

Roughly speaking, the correct result is that *as $h \to 0$, the solution becomes more and more symmetric about the point where the maximum is attained, and that point approaches the origin.* Thus, the correct result is that for a sufficiently small space step, the solution will be "approximately" symmetric about the origin.

This is one example of the type of general picture we have in mind: corresponding to every major theorem in the analytic or continuous setting, there should be an understanding of what the implications of this result are in the discrete approximating setting.

Naturally, these results have analogues, (although a bit more complicated to write down in a small number of pages), in the partial differential equation setting.

Similarly, there is a vast literature on when weak solutions of

$$\Delta u = f(u), \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{3}$$

are a priori bounded , with appropriate assumptions on f. For a complete list of references to this problem, see [2]

In [1], we show that these results must take account of sharp corners in the domain, resulting in a new class of generalised Brezis-Turner critical exponents. What is now under investigation is what the implications of this are for the finite difference approximations of the approximate solutions. When can we be sure that there will be uniform a priori bounds for finite difference approximations, at least when h is sufficiently small.

It is known that if the critical growth rates are exceeded in the nonlinearity, new singular solutions can exist. Presumably these will show up when one tries to find finite difference solutions of these equations. A natural question is how to distinguish these solutions from the classical ones.

References

[1] P. J. McKenna and W. Reichel, "A priori bounds for semilinear equations and a new class of critical exponents for Lipschitz domains," *J. Funct. Anal.*., vol.244, pp.220–246, 2007.

[2] P. J. McKenna and W. Reichel, "Gidas-Ni-Nirenberg results for finite difference equations:Estimates of approximate symmetry," *J.M.A.A.*, vol. 334, pp.206–222, 2007.