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A Time-Frequency method for Chaotic Flow

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Abstract—Time-frequency analysis is performed for chaotic flow with a power spectra estimator based on the phase-space neighborhood. It is observed that the nearest neighbors, representing the state recurrences in the phase space reconstructed by time delay embedding, actually cover data segments with similar wave forms and thus possess redundant information, but recur with no obvious temporal regularity. To utilize this redundant recurrence information, a neighborhood-based spectra estimator is devised. Then time-frequency analysis with this estimator is performed for the noisy Lorenz time series and colored noise. Features revealed by the spectrogram can be used to distinguish noisy chaotic flow from colored noise.

1. Introduction

In order to obtain the inherent properties of a chaotic system from the measured time series, various methods have been proposed, such as surrogate tests [1], Fourier transforms [2, 3], and approaches based on time delay embedding [9]. Among these methods, approaches based on time delay embedding may be the most popular framework for analyzing chaotic time series. Some measures such as Lyapunov exponents and correlation dimension have been proposed to characterize the global features of dynamical systems. However, few studies of the local time pattern of chaotic time series have been reported; nevertheless, this is important for some purposes, such as to reveal the degree of chaoticity of a sequence.

Spectra analysis provides an alternative framework for chaotic time series analysis [2, 3, 4]. With the methods based on Fourier transform, the relation between the spectra and the topology as the corresponding dynamical system bifurcates to chaos has been studied [2]. And the exponential-law fall-off pattern of some typical chaotic data (e.g., the Lorenz time series) has been utilized to distinguish chaotic sequence from colored noise with power-law spectra [4]. However, other researchers have argued that a chaotic sequence cannot be well distinguished from either colored noise [5] or quasi-periodic motion (with singular power spectra) by its finite-time power spectra [3]. This is especially true when the chaotic data are contaminated by observational noise. For a chaotic signal with complicated evolution, the simple frequency domain representation may obscure information related to timing. Spectra analysis usually only adopts the spectral amplitude, while neglecting the phase information. Consequently, confusion will occur between any two signals with the same spectral amplitudes. A time-frequency joint analysis is therefore desirable to better unveil these features [11]. However, few studies of the time-frequency analysis for chaotic sequence have been reported.

State recurrence is one important feature of chaotic systems. In the reconstructed phase space, the state recurrences of a reference phase point turn out to be its nearest neighbors, which can provide redundant information but recur with no temporal regularity as we will demonstrate later. The conventional time-frequency analysis methods (e.g., periodogram [10]), utilizes only one segment of consecutive data and neglects temporally isolated state recurrences beside this data segment. So a time-frequency analysis which can utilizes all state recurrences is desirable. The present paper focuses on: (i) demonstrating that the nearest neighbors can provide redundant information for chaotic signal analysis and processing, (ii) proposing a spectra estimator which can utilize all the neighbors, and (iii) performing a time-frequency analysis to (noisy) chaotic flow with the proposed spectra estimator and extracting some features that can be used to distinguish the (noisy) chaotic data from colored noise.

The organization of this paper is as follows. In Sec. 2, the relationship between the reference point and its nearest neighbors is demonstrated, and the principle of neighborhood-based spectra estimation (NSE) is presented. In Sec. 3, time-frequency analysis with NSE is performed for the (noisy) Lorenz time series and colored noise. It is shown that colored noise can be distinguished from (noisy) chaotic flow based on their respective main ridge patterns. Finally, a conclusion is given in Sec. 4.

2. Principle of the method

Let $\{z_n\}_{n=1}^{L}$ denote a chaotic time series with L samples. The phase points can be reconstructed by time delay embedding, i.e., $\{\mathbf{z}_n\}_{n=1}^{L-(d-1)\tau}$, $\mathbf{z}_n = [z_n, z_{n+\tau}, \cdots, z_{n+(d-1)\tau}]^{\mathrm{T}}$, where d is embedding dimension, τ is time delay, and $(\cdot)^{\mathrm{T}}$ denotes the transpose of a real matrix. The near neighborhood of the reference point $\mathbf{z}_n$ is defined as $\mathbf{N}_n \triangleq \{\mathbf{z}_k : \|\mathbf{z}_k - \mathbf{z}_n\| < \varepsilon, 1 \leq k \leq L-(d-1)\tau\}$, and arranged as $\mathbf{N}_n = \{\mathbf{z}_{k_1}, \mathbf{z}_{k_2}, \cdots, \mathbf{z}_{k_N}\}, k_1 < k_2 < \cdots < k_N$, where $N = |\mathbf{N}_n|$ is the number of neighbors, and ε is neighborhood radius (note that $\mathbf{z}_n \in \mathbf{N}_n$). Furthermore, the recurrence time of $\mathbf{z}_n$ is simply defined as $T_n(i) = k_{i+1} - k_i,\ i = 1, \cdots, N-1$ [12].

Considering a chaotic time series generated from the

Lorenz system [15], $\dot{x} = \sigma(y-x), \dot{y} = (r-z)x-y, \dot{z} = xy-bz$, where $(\sigma, r, b) = (10, 28, 8/3)$. Note that all the Lorenz time series used in this paper are 10 000 points sampled from the x component with time interval 0.04 and these fixed parameters, unless stated otherwise.

Figure 1 demonstrates the relationship between the reference point $\mathbf{z}_{2963}$ (randomly selected) and its first ten nearest neighbors with subscript $k = 192, 2659, 3485, 4387, 4388, 5376, 5415, 6763, 6764, 7235$. The reference point $\mathbf{z}_n$ covers a segment of time series $[z_n, z_{n+1}, z_{n+2}, \cdots, z_{n+(d-1)\tau}]^{\mathrm{T}}$ with the length of embedding window $L_w = (d-1)\tau + 1$. For clarity, let $\mathbf{s}_n$ denote this associated segment of the time series. If $\tau = 1$, $\mathbf{s}_n$ is the same as $\mathbf{z}_n$. It can be observed that the corresponding wave forms of the neighbors are similar to each other, but the recurrence time seems irregular. From the viewpoint of signal processing, these similar wave-form segments contain much redundant information relative to the reference one. There are some neighbors that are adjacent in time, for example $k = 4387$ and 4388. The adjacent neighbors that lie on the same recurrence trajectory provide only one new sample, primarily they serve to increase the weight of the corresponding state recurrence within the neighborhood.

Conventional linear techniques (e.g., classical Fourier transform) neglect some scattered state recurrences and just utilize one segment of consecutive data. Consequently, these techniques usually obtain poor results in analyzing chaotic data, while some methods (e.g., local projection noise reduction [6, 7] and nonlinear prediction [13]), specifically designed for chaotic data, utilize the neighbors and thus achieve better results. Analogously, aiming to use the state recurrence of a chaotic system, we propose a Neighborhood based Spectra Estimator (NSE) to estimate the corresponding power spectra of the reference phase point [8].

For the neighborhood $\mathbf{N}_n$, we define a $L_w \times N$ neighborhood matrix as $\mathbf{D}_n = [\mathbf{x}_{k_1} \ \mathbf{x}_{k_2} \ \cdots \ \mathbf{x}_{k_N}]$, with notation $\mathbf{x}_{k_i} = \mathbf{s}_{k_i} - \overline{\mathbf{s}}_n$, where $\overline{\mathbf{s}}_n = \langle \mathbf{s}_{k_i} \rangle$ is the center. First, an eigenvalue decomposition to the covariance matrix, i.e., $\mathbf{C}_n = \frac{1}{N}\mathbf{D}_n\mathbf{D}_n^{\mathrm{T}}$, of the neighborhood $\mathbf{N}_n$ is performed, $\mathbf{C}_n\mathbf{u}_i - \lambda_i\mathbf{u}_i = 0$, where λ_i is the i-th eigenvalue, and $\mathbf{u}_i = [u_i(1), u_i(2), \cdots, u_i(L_w)]^{\mathrm{T}}$ is its associated eigenvector. Then with the discrete-time Fourier transform of eigenvector $\mathbf{u}_i$,

$$V_i(\omega) = \sum_{p=1}^{L_w} u_i(p) e^{-jwp}, \tag{1}$$

NSE can be expressed as

$$P_{NSE}(\omega) = \frac{1}{L_w} \sum_{i=1}^{L_w} \lambda_i |V_i(\omega)|^2. \tag{2}$$

NSE is derived from the B-T spectra estimator [10]. The difference is that the B-T estimator utilizes the covariance matrix generated from only one segment of consecutive data, while NSE uses the covariance matrix estimated from the data segments covered by the temporally scattered nearest neighbors. Thus, NSE can capture the long time state recurrence of chaotic data. If the neighborhood contains only the reference point, NSE reduces to the B-T estimator.

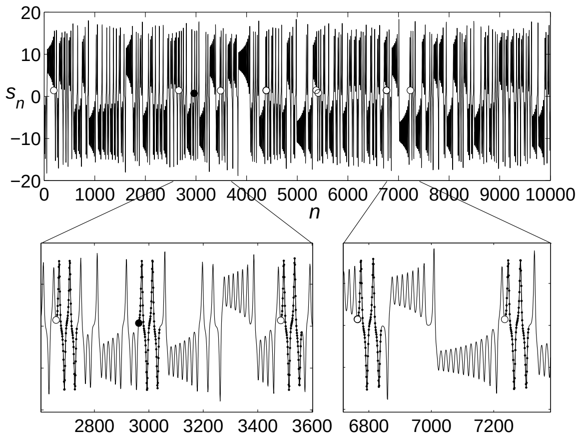

Figure 1: The Lorenz time series and the first ten nearest neighbors of reference point $\mathbf{z}_{2963}$. • — the sample z_{2963}; ◦ — the samples z_k corresponding to the first 10 nearest neighbors. The bottom panels are enlargements of short segments. Each segment marked with small dots corresponds to one neighbor in phase space.

Furthermore, for each reference point, we define the *main frequency* ω_m as

$$P(\omega_m) = \max P(\omega_l), \ \omega_m \in \{\omega_l\}, \tag{3}$$

where ω_l is the frequencies with local maximum power amplitude, i.e.

$$\frac{dP(\omega)}{d\omega}\bigg|_{\omega=\omega_l} = 0, \quad \frac{d^2P(\omega)}{d\omega^2}\bigg|_{\omega=\omega_l} < 0. \tag{4}$$

Then the *main frequency* will form a *main ridge* as the reference point moves along the phase trajectory. We observe in the following sections that this main ridge shows different characteristic patterns for different types of data.

3. Time-frequency analysis to chaotic time series

The Lorenz system is a typical chaotic system with two scrolls. Figure 2 shows the power specta of the Lorenz system estimated by periodogram. The power spectra of the x and y components are broadband and similar to each other, while the power spectra of the z component have a peak. This spectra peak, which is indicated by $\downarrow F_1$ in Fig. 2(c) and named the *hidden frequency* in references [14], can reveal the frequency related to the principal oscillation of the Lorenz system. This frequency is not a particular case of this sequence. The spectra peak universally exists with small deviation (1.305∼1.330 Hz with 95% confidence). Though this oscillation exists in the x and y components simultaneously as the dynamics evolves, the periodogram spectra of x and y fail to reveal it.

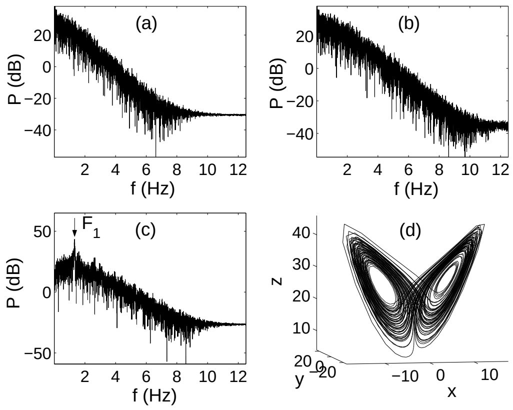

Figure 2: Power spectra of the Lorenz system estimated by periodogram. The y-axis label P denotes the power spectra of each sequence. (a), (b), and (c) are the power spectra of the time series measured from the x, y, and z components, respectively; (d) the strange attractor of the Lorenz system.

In contrast, time-frequency analysis of the same Lorenz time series used in Fig. 2(a) with NSE can reveal the principal oscillation. We over-embed the time series with $\tau = 4$ and $d = 20$, and use the first 20 nearest neighbors in NSE analysis. A 1000-sample segment of the spectrogram for each case is illustrated in Fig. 3. We can observe that the main ridge is formed by many short disjointed curves (even for the Lorenz time series contaminated with 5 dB white noise), which vary slowly around a frequency related to the principal oscillation. The frequency indicated by $\uparrow F_2$, is approximately equal to the hidden frequency indicated by $\downarrow F_1$ in Fig 2(c). This implies that the main frequencies contain the information of the principal oscillation of chaotic system. Similar results can also be obtained with data measured from the y component of the Lorenz system.

Similar wave forms covered by the neighbors can enhance their common structure, i.e., the principal oscillation, while may simultaneously "average" out the substructures and noise. Thus, even for the noisy Lorenz time series with 5 dB white noise, the principal oscillation can be extracted. The results show that time-frequency analysis with NSE can extract the hidden frequency appropriately (estimated from main frequencies like the spectra peak $\uparrow F_2$ in Fig. 3).

It is difficult to distinguish (noisy) chaotic data from colored noise by their spectra falloff pattern [5, 3]. Chaotic flow has scattered state recurrences, while colored noise does not possess this deterministic feature. Here, we will demonstrate that time-frequency analysis with NSE can reveal this difference, and thus can be an alternative method to distinguish them.

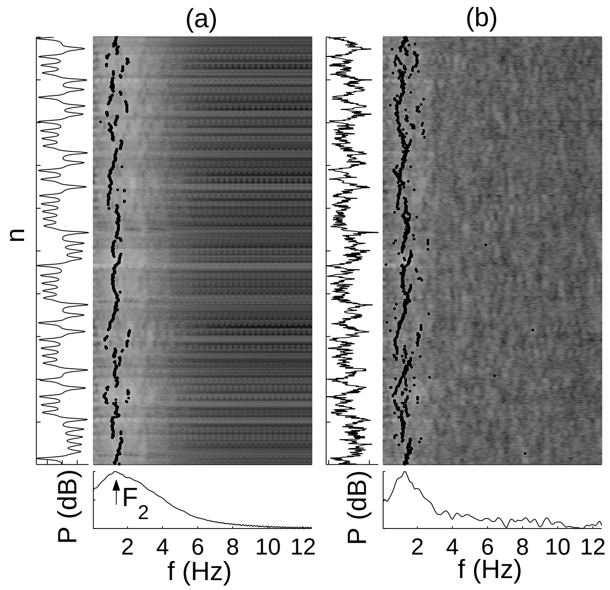

Figure 3: Spectrogram of the Lorenz time series, $r = 28$. (a) the clean Lorenz time series; (b) the noisy Lorenz time series with 5 dB additive white noise. For each sub-figure, the left panel is the time series, the top-right panel is the corresponding spectrogram estimated by NSE, and the bottom panel is the average of the spectrogram over time. The black points are the main frequencies. In this paper, Eq. 1 is implemented by 512-point fast Fourier transform (FFT) and $(512-L_w)$ zeros are padded to the end of $\mathbf{u}_i$ as the common strategy adopted. This pattern is followed in all following spectrogram figures, unless otherwise stated.

We take a pink noise and a surrogate sequence as examples. The pink noise (10 000 points) is generated by a special case of 1-order autoregressive process (AR(1)), $X_{n+1} = \beta X_n + (1-\beta)\epsilon_n$, where $\beta = 0.69$ and $\epsilon_n \sim N(0,1)$ is a Gaussian process. The spectra of the clean Lorenz data have a long exponential-law scaling region. As the Lorenz data are contaminated by observational noise, the exponential-law region becomes less obvious and difficult to be distinguished from that of pink noise. While the time-frequency analysis with NSE is sensitive to this difference. As Fig. 4(a) indicates, the main frequency of pink noise varies along time with no regularity, while the main ridge pattern of the noisy chaotic data with 5 dB white noise (Fig. 3(b)) exhibits more long-term temporal structure. The surrogate data are generated by shuffling the phase of the original noisy Lorenz data used in Fig. 3(b) [1]. The power spectra of the surrogate data are similar to that of the original data. However, due to the phase shuffling, the surrogate data do not possess the deterministic features of the original noisy Lorenz data, and thus their main ridge pattern is clearly distinct (Fig. 4(b) vs. Fig. 3(b)). As we discussed in Sec. 1, time-frequency joint analysis can reveal some information that is obscured by just a single finite time frequency representation.

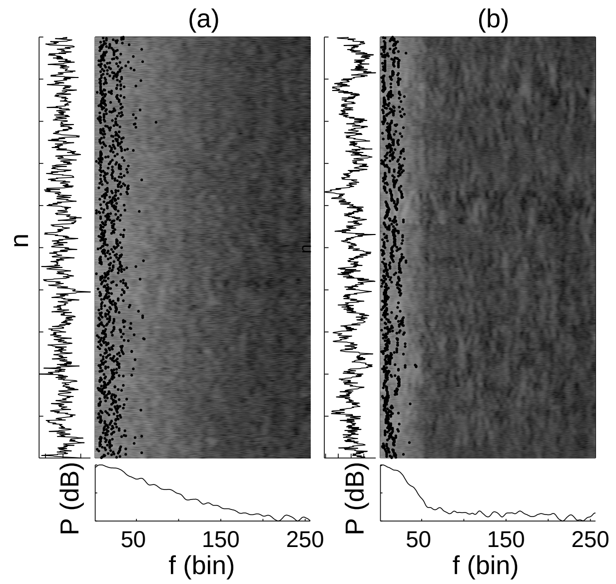

Figure 4: (a) Spectrogram of pink noise; (b) spectrogram of the surrogate data. The frequency bins calculated by FFT are not scaled to the real frequency with units of Hz.

In summary, for the clean chaotic Lorenz time series, the main ridge has many short unconnected curves, which vary around the hidden frequency. For the noisy chaotic flow, the principal oscillation can be extracted with the nearest neighbors, and thus the main ridge reserves some characteristics of the corresponding clean data, while for the pink noise and surrogate data, there is no deterministic feature, and thus the main ridge is irregular, which is distinct from that of (noisy) chaotic flow. The difference in main ridge pattern can be used to distinguish them.

4. Conclusion

First, we over-embedded the chaotic data, and demonstrated that the nearest neighbors represent the state recurrences of the reference point, but recur with no obvious temporal regularity. To apply these state recurrences, a neighborhood based spectra estimator (NSE) was devised for chaotic flow, bridging time delay embedding and the frequency domain. Then time-frequency analysis with NSE was performed for (noisy) Lorenz time series. We found that NSE can reveal the frequency related to the principal oscillation of the dynamical system, which is hidden in the spectra estimated by the periodogram method. Further we applied NSE to pink noise and phase shuffled surrogate data. The results show that their main ridge patterns are distinct from that of (noisy) chaotic flow, thus providing an alternative method to distinguish colored noise from (noisy) chaotic flow, though for some real or more chaotic systems, a distinction may not be that easy.

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