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A class of ill-conditioned nonlinear algebraic equations

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Abstract—This paper studies a kind of ill-conditioned algebraic equations, which have very close two ill-conditioned solutions. The equations are related to nonlinear resistive networks.

1. Introduction

Rump[1] derived extraordinarily ill-conditioned linear simultaneous equations under the floating-point arithmetic, where condition numbers may be larger than 10^{50}. These equations are very useful to examine the quality of accuracy guaranteed algorithms to solve linear equations[2]. To examine the quality of various accuracy guaranteed algorithms for solving nonlinear equations, it is desirable to generate other kinds of ill-conditioned equations. In this paper we study on a very special kind of nonlinear equations, which are derived from a transistor circuit and for which the conditions of a globally unique solution[3]–[7] and the number of solutions[8]–[11] have been investigated for a long time. We show that they may possess very closely located ill-conditioned solutions, which means that the Jacobian matrices of these solutions have fairly large condition numbers. We suppose that the values treated are real and we consider moderately (but not extraordinarily) ill-conditioned equations, which means the condition number of $10^5 \sim 10^{10}$.

A typical equation difficult to solve accurately is an equation with multiple solutions such as $(x-1)^{10}=0$. But the variety of them is small. The purpose of this paper is to generate easily many equations difficult to solve.

2. Equations

Circuit equations of a transistor circuit, where each transistor is represented by the Ebers-Moll models, are written as:

$$\begin{bmatrix} \alpha_1 e^{\beta_1 x_1} \\ \alpha_2 e^{\beta_2 x_2} \\ \vdots \\ \alpha_n e^{\beta_1 x_n} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \tag{1}$$

where the parameters a_{ij} and b_i are determined by the values of resistors and voltage/currrent sources, and therefore are restricted to some physical (circuit-theoretic) constraints. In this paper however we ignore these physical restrictions and assume their parameters as arbitrary real values.

Without loss of generality we can assume that Eq. (1) has a solution x_* $[x_{*1}, \cdots, x_{*n}]^T = [0, \cdots, 0]^T$. where the superscript T means the transpose. This is easily done by choosing b_i appropriately. The Jacobian matrix J of Eq.(1) at the solution x_* is given as:

$$\begin{aligned} J &= \operatorname{diag}\left[\alpha_1\beta_1 e^{\beta_1 x_{*1}} \quad \alpha_2\beta_2 e^{\beta_2 x_{*2}} \quad \cdots \quad \alpha_n\beta_n e^{\beta_n x_{*n}} \right] \\ &+ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \end{aligned} \tag{2}$$

Let an $n \times n$ matrix $P = [p_{ij}]$ have a large condition number such as $10^5 \sim 10^{10}$ and let

$$J = P. \tag{3}$$

Instead of Eq.(1), we consider the following equations:

$$\left.\begin{array}{l} \gamma x_1^2 + p_{11}x_1 + p_{12}x_2 + p_{13}x_3 + \cdots + p_{1n}x_n = 0 \\ \gamma x_2^2 + p_{21}x_1 + p_{22}x_2 + p_{23}x_3 + \cdots + p_{2n}x_n = 0 \\ \cdots\cdots\cdots \\ \gamma x_n^2 + p_{n1}x_1 + p_{n2}x_2 + p_{n3}x_3 + \cdots + p_{nn}x_n = 0 \end{array}\right\} \tag{4}$$

which is derived by truncating the third and the higher order terms of the Taylor series of $\alpha_i e^{\beta_i x_i}$ in Eq.(1), where we assumed that all $\gamma_i \equiv \alpha_i \beta_i^2$ are the same value γ.

3. Case of $n = 2$

First we consider the simplest case of $n = 2$. Then

$$\gamma x_1^2 + p_{11}x_1 + p_{12}x_2 = 0 \tag{5}$$

$$\gamma x_2^2 + p_{21}x_1 + p_{22}x_2 = 0 \tag{6}$$

Let

$$\Delta \equiv |P| = p_{11}p_{22} - p_{12}p_{21} \tag{7}$$

Assumption 1:$|\Delta|$ is very small, i.e., $10^{-5} \sim 10^{-10}$
Assumption 2:$|\gamma|$ is of a moderate magnitude, i.e., $10^{-2} \sim 10^2$
Assumption 3: p_{ij} looks random numbers of the values $10^{-1} \sim 10^1$.

Then the condition number $\sum p_{ij}^2/|\Delta|$ is roughly $10^5 \sim 10^{10}$

The matrix P is, for example, like $P = \begin{bmatrix} 1 & 2 \\ 2 & 4.0001 \end{bmatrix}$.

Let

$$g_1 \equiv \gamma x_1^2 + p_{11}x_1 + p_{12}x_2 \tag{8}$$

$$g_2 \equiv \gamma x_2^2 + p_{21}x_1 + p_{22}x_2 \tag{9}$$

To solve Eqs.(5) and (6) is equivalent to solve

$$g_1 = 0, \quad g_2 = 0 \tag{10}$$

In the following we show that Eqs.(5)–(6) has a solution very close to the solution $x_* = 0$ under some mild assumptions like Assumptions 2 and 3. We will show it by three different ways. In any case we use the fixed point theorem for the proof; that is, we rewrite the equation as

$$x = G(x) \ \ (\equiv b_0 + b_2 x^2 + b_3 x^3 \cdots) \tag{11}$$

where $G(x)$ does not contain the first order term of x. If $b_0 > 0$, let $X = [\frac{1}{2}b_0, \frac{3}{2}b_0]$ and prove:
Condition A: $G(X) \subset X$
Condition B: There is a constant $0 < k < 1$ such that
$|G(x') - G(x'')| \leq k|x' - x''|$ for $x', x'' \in X$

3.1. Direct calculation (Method 1)

On the assumption that $p_{12} \neq 0$ Eq.(5) yields:

$$x_2 = -\frac{p_{11}x_1 + \gamma x_1^2}{p_{12}} \tag{12}$$

which is substituted into Eq.(6). Then we have:

$$p_{21}x_1 - p_{22}\frac{p_{11}x_1 + \gamma x_1^2}{p_{12}} + \gamma\left\{\frac{p_{11}x_1 + \gamma x_1^2}{p_{12}}\right\}^2 = 0 \tag{13}$$

from which we have:

$$\gamma^3 x_1^3 + 2\gamma^2 p_{11} x_1^2 + \gamma(p_{11}^2 - p_{12}p_{22})x_1 - p_{12}\Delta = 0 \tag{14}$$

If we assume that the coefficient of x_1 in Eq.(14) has a moderate magnitude ($10^{-2} \sim 10^2$), we have from Eq.(14)

$$x_1 \approx \frac{p_{12}}{\gamma(p_{11}^2 - p_{12}p_{22})}\Delta \ \ (\equiv x_{10}) \tag{15}$$

Substituting this into Eq.(12), we have:

$$x_2 \approx \frac{p_{12}^2 p_{21} - p_{11}^3}{\gamma(p_{12}p_{22} - p_{11}^2)^2}\Delta \ \ (\equiv x_{20}) \tag{16}$$

Thus we have a candidate of an approximate solution (x_{10}, x_{20}) near the origin. We have to show rigorously that there exists a solution. Eq. (14) can be rewritten as:

$$x_1 = x_{10} + \frac{\gamma p_{11} x_1^2}{p_{12}p_{22} - p_{11}^2} + \frac{\gamma^2 x_1^3}{p_{12}p_{22} - p_{11}^2} (\equiv G(x_1)) \tag{17}$$

For $G(x_1)$ we can easily verify that Conditions A and B above hold if $|\Delta|$ is sufficiently small, i.e., if $|\Delta| < \left|\frac{(p_{12}p_{22} - p_{11}^2)^2}{2\gamma p_{11} p_{12}}\right|$. This condition is usually satisfied on Assumptions 2 and 3.

This direct method is easy to verify a solution, but can not be generalized to the case of $n > 2$.

3.2. Taylor series expansion of g_2 with respect to x_1 (Method 2)

In this case we regard x_2 as a function of x_1 satisfying Eq. (5) and we regard the function g_2 in Eq. (9) as a function of x_1. To investigate the property of $g_2(x_1)$ in Eq. (9) we expand it around the origin $x_1 = 0$. By noting $g_2(0) = 0$, $g_2(x_1)$ can be expanded in the form:

$$g_2(x_1) \equiv \frac{1}{1!}\left.\frac{dg_2}{dx_1}\right|_{x_1=0} x_1 + \frac{1}{2!}\left.\frac{d^2 g_2}{dx_1^2}\right|_{x_1=0} x_1^2 + \frac{1}{3!}\left.\frac{d^3 g_3}{dx_1^3}\right|_{x_1=0} x_1^3 + \cdots \tag{18}$$

where

$$g_2' = \frac{dg_2}{dx_1} = 2\gamma x_2 x_2' + p_{21} + p_{22}x_2' \tag{19}$$

$$g_2'' = \frac{d^2 g_2}{dx_1^2} = 2\gamma[(x_2')^2 + x_2 x_2''] + p_{22}x_2'' \tag{20}$$

$$g_2''' = \frac{d^3 g_2}{dx_1^3} = 2\gamma[3x_2' x_2'' + x_2 x_2'''] + p_{22}x_2''' \tag{21}$$

$$g_2^{(4)} = \frac{d^4 g_2}{dx_1^4} = 2\gamma[3(x_2'')^2 + 4x_2 x_2''' + x_2 x_2^{(4)}] + p_{22}x_2^{(4)} \tag{22}$$

To obtain $x_2^{(k)}$ $(k = 1, 2, \cdots)$ in Eqs.(19)–(22) we differentiate Eq.(5) successively with respect to x_1 as follows:

$$2\gamma x_1 + p_{11} + p_{12}x_2' = 0 \tag{23}$$

$$2\gamma + p_{12}x_2'' = 0 \tag{24}$$

$$x_2^{(k)} = 0 \ \ (k = 3, 4, \cdots,) \tag{25}$$

By letting $x_1 = 0$ in the above, we have:

$$x_2'\big|_{x_1=0} = -\frac{p_{11}}{p_{12}}, \quad x_2''\big|_{x_1=0} = -\frac{2\gamma}{p_{12}} \tag{26}$$

$$x_2^{(k)}\big|_{x_1=0} = 0 \ \ (k = 3, 4, \cdots,) \tag{27}$$

Substituting Eqs.(26) and (27) into Eqs.(19), we have

$$g_2'\big|_{x=0} = p_{21} - p_{22}\frac{p_{11}}{p_{12}} = -\frac{\Delta}{p_{12}} \tag{28}$$

$$g_2''\big|_{x=0} = 2\gamma\left(-\frac{p_{11}}{p_{12}}\right)^2 + p_{22}\left(-\frac{2\gamma}{p_{12}}\right) = 2\gamma\frac{p_{11}^2 - p_{12}p_{22}}{p_{12}^2} \tag{29}$$

$$g_2'''\big|_{x=0} = 2\gamma\left[3\left(-\frac{p_{11}}{p_{12}}\right)\left(-\frac{2\gamma}{p_{12}}\right)\right] = 2\gamma^2\frac{p_{11}}{p_{12}^2} \tag{30}$$

$$g_2^{(4)}\big|_{x=0} = 2\gamma\left[3\left(-\frac{2\gamma}{p_{12}}\right)^2\right] = 24\gamma^3\frac{1}{p_{12}^2} \tag{31}$$

$$g_2^{(k)}\big|_{x=0} = 0 \ \ (k = 5, 6, \cdots,) \tag{32}$$

Thus the Taylor expansion ends in a finite terms (and $g_2(x_1)$ is a polynomial of order 4, which is the same as Eq. (14) except for a constant multiplier. Thus this method is essentially the same as the direct method.

We stated Method 2 in connection to Method 3 below.

3.3. Taylor series expansion of g_1 with respect to x_1 (Method 3)

In this case we regard x_2 as a function of x_1 satisfying Eq. (6) and we regard the function g_1 in Eq. (8) as a function of x_1. To investigate the property of $g_1(x_1)$ we expand it in the Taylor series. For simplicity we denote g_1 by g.

$$g(x_1) \equiv \frac{1}{1!}\left.\frac{dg}{dx_1}\right|_{x_1=0} x_1 + \frac{1}{2!}\left.\frac{d^2g}{dx_1^2}\right|_{x_1=0} x_1^2 + \frac{1}{3!}\left.\frac{d^3g}{dx_1^3}\right|_{x_1=0} x_1^3 + \cdots \quad (33)$$

As for Eq.(33), we have by Eq.(8):

$$g' = \frac{dg}{dx_1} = 2\gamma_1 x_1 + p_{11} + p_{12}x_2' \quad (34)$$

$$g'' = \frac{d^2g}{dx_1^2} = 2\gamma_1 + p_{12}x_2'' \quad (35)$$

$$g^{(k)} = \frac{d^kg}{dx_1^k} = p_{12}x_2^{(k)}(k = 3, 4, \cdots,) \quad (36)$$

So we have to calculate $x_2^{(k)}$ $(k = 1, 2, \cdots)$ in Eq.(34)–(36). Differentiating Eq.(6),

$$\gamma(x_2'x_2 + x_2x_2') + p_{21} + p_{22}x_2' = 0 \quad (37)$$

Eq.(37) can be rewritten by using the notation $\left(\begin{array}{c} m \\ h \end{array}\right) = m!/h!(m-h)!$ as follows;

$$\gamma\left[\left(\begin{array}{c} 1 \\ 0 \end{array}\right)x_2'x_2 + \left(\begin{array}{c} 1 \\ 1 \end{array}\right)x_2x_2'\right] + p_{21} + p_{22}x_2' = 0 \quad (38)$$

Similarly by differentiating Eq.(38) successively we have:

$$\gamma\left[\left(\begin{array}{c} 2 \\ 0 \end{array}\right)x_2''x_2 + \left(\begin{array}{c} 2 \\ 1 \end{array}\right)x_2'x_2' + \left(\begin{array}{c} 2 \\ 2 \end{array}\right)x_2x_2''\right] + p_{22}x_2'' = 0 \quad (39)$$

Generally we have

$$\gamma\left[\sum_{h=0}^{k}\left(\begin{array}{c} k-h \\ h \end{array}\right)x_2^{(k-h)}x_2^{(h)}\right] + p_{22}x_2^{(k)} = 0 \quad (40)$$

By using these formula, we can calculate the values $x_2^{(k)}$ at $x_1 = 0$ as follows:

$$x_2' = -\frac{p_{21}}{p_{22}}, \quad x_2'' = -\frac{\gamma p_{21}^2}{p_{22}^3} \times 2 \quad (41)$$

$$x_2''' = -\frac{\gamma^2 p_{21}^3}{p_{22}^5} \times 2 \times 3! = -\frac{\gamma^2 p_{21}^3}{p_{22}^5} \times 12 \quad (42)$$

$$x_2^{(4)} = -\frac{\gamma^3 p_{21}^4}{p_{22}^7} \times 5 \times 4! = -\frac{\gamma^3 p_{21}^4}{p_{22}^7} \times 120 \quad (43)$$

Continuing the similar calculation, we generally have:

$$x_2^{(k)} = -\frac{\gamma^{k-1}p_{21}^k}{p_{22}^{2k-1}} \times \delta_k \times k! \quad (44)$$

where δ_k obeys the following nonlinear recursive formula as follows:

$$\delta_1 = 1, \quad \delta_2 = 1, \quad \delta_k = \sum_{h=1}^{k-1}\delta_{k-h}\delta_h \quad (45)$$

Note that no cancellation occurs in the above calculation.

For the first several δ_k $(k = 3, 4, \cdots,)$ we have:

$$\delta_3 = 2, \ \delta_4 = 5, \ \delta_5 = 12, \ \delta_6 = 42, \ \delta_7 = 132$$
$$\delta_8 = 429, \ \delta_9 = 1430, \ \delta_{10} = 4862 \quad (46)$$

Thus δ_k increases very rapidly. We have

1. δ_k/δ_{k-1} increases with k.
2. $\lim_{k\to\infty}\delta_k/\delta_{k-1} = 4$

Unfortunately the authors do not have the rigorous proof of the above but verified it numerically.

From the above we have:

$$|x_2^{(k)}| \leq \frac{\gamma^{k-1}p_{21}^k}{p_{22}^{2k-1}} \times 4^{k-1} \times k! \quad (47)$$

Then we have

$$g(x_1) = g_1x_1 + g_2x_2^2 + g_3x_2^3 + \cdots = x_1(g_1 + g_2x_2 + g_3x_2^2 + \cdots) \quad (48)$$

where

$$g_1 = p_{11} + p_{12}\left.\frac{dx_2}{dx_1}\right|_{x_1=0} = \frac{\Delta}{p_{22}} \quad (49)$$

$$g_2 = \frac{1}{2!}\left(2\gamma + p_{12}\left.\frac{d^2x_2}{dx_1^2}\right|_{x_1=0}\right) = \gamma\left[1 - p_{12}\left(-\frac{p_{21}^2}{p_{22}^3}\right)\right] \quad (50)$$

$$g_k = p_{12}\left(-\frac{\gamma^{k-1}p_{21}^k}{p_{22}^{2k-1}}\right)\delta_k \quad (k = 3, 4, \cdots) \quad (51)$$

Suppose that $|p_{22}|$ has a moderate magnitute. Then from Eq.(49) $\left.\frac{dg}{dx_1}\right|$ has a magnitude of Δ.

Due to Eq.(51) we see that *Eq.(48) converges for* $|x_1|$ *small* $(|x_1| < 4p_{22}^2/\gamma p_{21})$. Let

$$G(x_1) \equiv -\frac{g_1}{g_2} - \frac{g_3}{g_2}x_1^2 - \frac{g_4}{g_2}x_1^3 - \frac{g_5}{g_2}x_1^4 - \cdots \quad (52)$$

Then a solution of $g(x_1) = 0$ is given as a solution of

$$x_1 = G(x_1) \quad (53)$$

Since $|\Delta|$ is very small, we guess from Eq.(49) that

$$x_1 \approx -g_1/g_2 \ \ (\equiv x_{10}) \quad (54)$$

may be its solution. Conditions A and B are easily verified for Eq.(54)

4. General case of $n > 2$

For simplicity we first discuss the case of $n = 4$ in Eq.(4) instead of a general n. We use a similar method as Method 3 in the above. For simplicity let

$$\tilde{x} \equiv [x_2, x_3, x_4]^T, \quad \tilde{p}_{1\cdot} \equiv [p_{12}, p_{13}, p_{14}]^T, \quad \tilde{p}_{\cdot 1} \equiv [p_{21}, p_{31}, p_{41}]^T$$

and let $\tilde{P}$ be an 3×3 matrix obtained from P by deleting the first row and the first column. Let

$$g(x_1) \equiv \gamma x_1^2 + p_{11}x_1 + \tilde{p}_{1\cdot}.\tilde{x} \quad (55)$$

Then we have

$$g' = 2\gamma x_1 + p_{11} + \tilde{p}_{1\cdot}.\tilde{x}' \quad (56)$$
$$g'' = 2\gamma + \tilde{p}_{1\cdot}.\tilde{x}'' \quad (57)$$
$$g''' = \tilde{p}_{1\cdot}.\tilde{x}''' \quad (58)$$
$$g^{(4)} = \tilde{p}_{1\cdot}.\tilde{x}^{(4)} \quad (59)$$

$\tilde{x}^{(k)}(k = 1, 2, \cdots)$ can be calculated by differentiating successively

$$\gamma\tilde{x}^2 + p_{\cdot 1}x_1 + \tilde{P}\tilde{x} = 0 \quad (60)$$

with respect to x_1 where $\tilde{x}^2$ means $[x_2^2, x_3^2, x_4^2]^T$. That is,

$$\gamma\frac{d}{dx_1}\left\{\tilde{x}^2\right\} + p_{\cdot 1} + \tilde{P}\tilde{x}' = 0 \quad (61)$$

$$\gamma\frac{d^k}{dx_1^k}\left\{\tilde{x}^2\right\} + \tilde{P}\tilde{x}^{(k)} = 0 \quad (k = 2, 3, \cdots) \quad (62)$$

The above equations are formally almost the same as Eqs.(37)– (40). Therefore we can use a similar calculation as δ_k in Eqs.(44), (45), etc. However there is a great difference between Eqs. (62) and (40) since $\tilde{P}$ in (62) is a matrix but not a scalar. Due to this fact the expression of $\tilde{x}^{(k)}$ becomes very complex. To give its expression we define some notations: Let $\Delta\begin{pmatrix} i \\ j \end{pmatrix}$ be the minor of p_{ij} and $\Delta\begin{pmatrix} i_1 i_2 \\ j_1 j_2 \end{pmatrix}$ are similarly defined. Then for example,

$$x_2'|_{x_1=0} = -\Delta\begin{pmatrix} 1 \\ 2 \end{pmatrix}/\Delta\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (63)$$

$$x_2''|_{x_1=0} = -\frac{2\gamma}{\Delta\begin{pmatrix} 1 \\ 1 \end{pmatrix}^3} \times \left[\Delta\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\Delta\begin{pmatrix} 1 \\ 2 \end{pmatrix}^2 + \Delta\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}\Delta\begin{pmatrix} 1 \\ 3 \end{pmatrix}^2 + \Delta\begin{pmatrix} 1 & 4 \\ 1 & 2 \end{pmatrix}\Delta\begin{pmatrix} 1 \\ 4 \end{pmatrix}^2\right] \quad (64)$$

$$\tilde{p}_{1\cdot}\tilde{P}^{-1} = \left[\ \Delta\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \Delta\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \Delta\begin{pmatrix} 4 \\ 1 \end{pmatrix}\ \right] \quad (65)$$

$$g' = \Delta/\Delta\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (66)$$

$$g'' = 2\gamma + \frac{1}{\Delta\begin{pmatrix} 1 \\ 1 \end{pmatrix}^3} \times \left[\Delta\begin{pmatrix} 2 \\ 1 \end{pmatrix}\Delta\begin{pmatrix} 1 \\ 2 \end{pmatrix}^2 + \Delta\begin{pmatrix} 3 \\ 1 \end{pmatrix}\Delta\begin{pmatrix} 1 \\ 3 \end{pmatrix}^2 + \Delta\begin{pmatrix} 4 \\ 1 \end{pmatrix}\Delta\begin{pmatrix} 1 \\ 4 \end{pmatrix}^2\right] \quad (67)$$

From the process of calculation we see that the element of $x_k^{(k)}$ is of the form:

$$\gamma^{k-1}N/\Delta\begin{pmatrix} 1 \\ 1 \end{pmatrix}^{2k-1} \quad (68)$$

where N is sum of the product of k $\Delta\begin{pmatrix} 1 & i \\ 1 & j \end{pmatrix}(i, j = 2, 3, 4)$ and $\Delta\begin{pmatrix} 1 \\ j \end{pmatrix}(j = 2, 3, 4)$. However we cannot expect the cancellation among these terms since even for $n = 2$ case there occur no calculation.

Let the maximum values of $\Delta\begin{pmatrix} 1 & i \\ 1 & j \end{pmatrix}$, $\Delta\begin{pmatrix} 1 \\ j \end{pmatrix}$, and $\Delta\begin{pmatrix} i \\ 1 \end{pmatrix}$ be respectively M_1, M_2, and M_3. Then for general n we can estimate $|g_k|$ as follows (proof omitted):

$$|g_k| \leq [4(n-1)]^k\gamma^{k-1}M_3M_1^kM_2^k/\Delta\begin{pmatrix} 1 \\ 1 \end{pmatrix}^{2k-1} \quad (69)$$

Therefore $\sum_{k=0}^{\infty} g_k x_1^k$ converges for $|x_1|$ small. We see that the case $n = 2$ is a special case of the above result.

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Appendix: Generation of the matrix P

Let K be a real skew matrix. Then

$$T = (E - K)(E + K)^{-1}$$

is an orthogonal matrix.

We choose T_1 and T_2 by randomly generating skew matrices K and let P be a matrix

$$P = T_1\text{diag}[\lambda_1, \lambda_2, \cdots, \lambda_n]T_2$$

where $|\lambda_i|$ $(i = 1, \cdots, n-1)$ are slmost the same value and $|\lambda_n|$ is very small, i.e., 10^{-5}.

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