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Formal Linearization for Time-Delay Nonlinear Systems using Chebyshev Expansion

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Abstract—This paper is concerned with a linearization approach for nonlinear systems with delays in the state variables using Chebyshev expansion. This approach is based on coordinate transformation with respect to a linearization function which is defined by Chebyshev polynomials. The resulting time-delay linearized systems can be analytically obtained by using Chebyshev expansion and its inversion is simple. As a tentative application of this linearization approach, a time-delay nonlinear observer is synthesized. Numerical experiments are illustrated to show effectiveness of this linearization for time-delay nonlinear systems.

1. Introduction

Studies for linear systems with delays in the state variables have been developed and many theories were proposed(e.g. [1]). On the other hands, a few of them have been done for nonlinear systems with delays(e.g. [2, 3, 4]). We have studied formal linearization methods for nonlinear systems [5, 6, 7], and applied one of them to a time-delay nonlinear system using Taylor expansion [8].

In this paper we develop a formal linearization for time-delay nonlinear systems in order to improve the accuracy of linearization by exploiting Chebyshev expansion [9]. By introducing a linearization function that is composed of the Chebyshev polynomials and a time-delay operator, time-delay nonlinear systems are transformed into time-delay linear ones with respect to the linearization function by using Chebyshev expansion. Its inversion is easy to obtain because of having the original state values within the linearization function. As an application of this linearization, a nonlinear observer with delays is synthesized.

Numerical experiments are illustrated and indicate that the accuracy of the approximation by the linearization improves as the order of the Chebyshev polynomials increases.

2. Formal Linearization

For the sake of simplicity, we consider a formal linearization method for scalar systems. For vector systems, it is straightforward. A nonlinear system with delays in the state variable is given by

$$\Sigma_1 : \dot{x}(t) = f\big(x(t)\big) + g\big(x(t-\ell)\big) \tag{1}$$

where $t > 0$ denotes time, overdot represents derivative with respect to t, $\ell \in (0, \infty)$ is the system delay, x is a state variable, $f \in C^1$ and $g \in C^1$ are nonlinear functions. The system initial conditions are given by

$$x(\theta) = \varphi(\theta)\ (-\ell \le \theta < 0),$$
$$x(0) = x_0. \tag{2}$$

A formal linearization is based on Chebyshev expansion [9]. Since the basic domain of the Chebyshev polynomials is defined by

$$D_0 = [-1,\ 1], \tag{3}$$

x is changed into y by

$$y = \frac{x-m}{p} \in D_0 \tag{4}$$

where m is the middle point of the operating domain of x and p half the operating domain. From Eq.(4), the given nonlinear system(Eq.(1)) is expressed by

$$\dot{y}(t) = \frac{1}{p}\Big\{f\big(py(t)+m\big) + g\big(py(t-\ell)+m\big)\Big\}. \tag{5}$$

The Chebyshev polynomials are defined by

$$T_i(y) = \cos(i \cdot \cos^{-1} y) \tag{6}$$

or,

$$T_0(y) = 1,\ T_1(y) = y,\ T_2(y) = 2y^2 - 1,$$
$$T_3(y) = 4y^3 - 3y,\ T_4(y) = 8y^4 - 8y^2 + 1,$$
$$T_5(y) = 16y^5 - 20y^3 + 5y, \cdots.$$

The recurrence formula of the Chebyshev polynomials is described by

$$T_{i+1}(y) = 2yT_i(y) - T_{i-1}(y),\ (i \ge 1), \tag{7}$$

$$T_0(y) = 1,\ T_1(y) = y,$$

and the derivative of the Chebyshev polynomials :

$$S_i(y) \equiv \frac{dT_i(y)}{dy}$$

has the recurrence formula :

$$S_{i+1}(y) = 2T_i(y) + 2yS_i(y) - S_{i-1}(y),\ (i \geq 1), \tag{8}$$

$$S_0(y) = 0,\ S_1(y) = 1.$$

Using these Chebyshev polynomials, let us define an N-th order linearization function $\boldsymbol{\phi}(\cdot) = \boldsymbol{\phi}\big(y(\cdot)\big)$ by

$$\begin{aligned}\boldsymbol{\phi} &= [\phi_1, \phi_2, \cdots, \phi_i, \cdots, \phi_N]^T \\ &= [T_1(y), T_2(y), \cdots, T_i(y), \cdots, T_N(y)]^T\ . \end{aligned} \tag{9}$$

And let a time-delay operator δ be

$$\delta\{y(t)\} = y(t - \ell)\ . \tag{10}$$

From the linearization function and the time-delay operator, the derivative of each element of $\boldsymbol{\phi}$ along with solution of the given nonlinear system(Eq.(5)) is as follows:

$$\begin{aligned}\dot{\phi}_i(y(t)) &= \dot{T}_i(y(t)) \\ &= \frac{d}{dy}T_i(y(t))\dot{y}(t) = S_i(y(t))\dot{y}(t) \\ &= \frac{1}{p}S_i(y(t))\Big\{f\big(py(t) + m\big) + g\big(p\delta y(t) + m\big)\Big\}\ . \end{aligned} \tag{11}$$

Note that Chebyshev expansion up to the N-th order derives

$$\frac{1}{p}S_i(y(t))f\big(py(t) + m\big) = \sum_{j=0}^{N} a_{i\,j}T_j(y) + \text{higher order} \tag{12}$$

where

$$a_{i\,j} = \frac{2}{\pi}\int_{-1}^{1}\frac{\frac{1}{p}S_i(y(t))f\big(py(t) + m\big)T_j(y)}{\sqrt{1 - y^2}}dy \quad (j \neq 0),$$

$$a_{i\,0} = \frac{1}{\pi}\int_{-1}^{1}\frac{\frac{1}{p}S_i(y(t))f\big(py(t) + m\big)}{\sqrt{1 - y^2}}dy$$

and

$$\frac{1}{p}S_i(y(t))\Big(g\big(p\delta y(t) + m\big)\Big) = \sum_{j=0}^{N} b_{i\,j}(\delta)T_j(y) + \text{higher order} \tag{13}$$

where

$$b_{i\,j}(\delta) = \frac{2}{\pi}\int_{-1}^{1}\frac{\frac{1}{p}S_i(y(t))\Big(g\big(p\delta y(t) + m\big)\Big)T_j(y)}{\sqrt{1 - y^2}}dy \quad (j \neq 0),$$

$$b_{i\,0}(\delta) = \frac{1}{\pi}\int_{-1}^{1}\frac{\frac{1}{p}S_i(y(t))\Big(g\big(p\delta y(t) + m\big)\Big)}{\sqrt{1 - y^2}}dy\ .$$

From Eqs.(12) and (13), Eq.(11) is approximated by the Chebyshev polynomials:

$$\dot{\phi}_i(y) \approx \sum_{j=0}^{N} a_{i\,j}T_j(y) + \sum_{j=0}^{N} b_{i\,j}(\delta)T_j(y), \quad (i = 1, \cdots, N)$$

and $\dot{\boldsymbol{\phi}}$ is approximated by

$$\dot{\boldsymbol{\phi}}(y) = A(\delta)\boldsymbol{\phi}(y) + B(\delta) \tag{14}$$

where

$$A(\delta) = [a_{i\,j} + b_{i\,j}(\delta)] \in R^{N\times N}\ (i, j = 1, \cdots, N)\ ,$$

$$B(\delta) = [a_{i\,0} + b_{i\,0}(\delta)] \in R^{N}\ (i = 1, \cdots, N)\ .$$

Thus a formal time-delay linear system is derived by

$$\Sigma_2 : \dot{\boldsymbol{z}}(y) = A(\delta)\boldsymbol{z}(y) + B(\delta)\ , \tag{15}$$

$$\boldsymbol{z}\big(y(\theta)\big) = \boldsymbol{\phi}\Big(\frac{\varphi(\theta) - m}{p}\Big) \quad (-\ell \leq \theta < 0),$$

$$\boldsymbol{z}(0) = \boldsymbol{\phi}\Big(\frac{x_0 - m}{p}\Big)\ .$$

Its inversion is simply obtained as follows. From Eq.(9), an approximated value $\hat{x}(t)$ is

$$\hat{x}(t) = p[1\ 0\ \cdots\ 0]\boldsymbol{\phi}(y(t)) + m = p[1\ 0\ \cdots\ 0]\boldsymbol{z}(t) + m\ . \tag{16}$$

3. Time-Delay Nonlinear Observer

As a tentative application of this method, we synthesize a time-delay nonlinear observer. The system is the same as Eq.(1):

$$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t - \ell)\big) \tag{17}$$

and a measurement equation is assumed to be

$$\eta(t) = C(\delta)\boldsymbol{\phi} \in R^N \tag{18}$$

where $C(\delta) \in R^{N\times N}$ could be a matrix function with respect to the time-delay operator δ.

The time-delay nonlinear system (Eq.(17)) is transformed into the time-delay linear one (Eq.(15)) by the formal linearization as mentioned above. By applying the linear observer theory [10] to these time-delay linear systems (Eqs.(15) and (18)), an observer is obtained as follows:

$$\dot{\hat{\boldsymbol{z}}}(t) = A(\delta)\hat{\boldsymbol{z}}(t) + L(\eta - \hat{\eta}) + B(\delta), \tag{19}$$

$$\hat{\eta} = C(\delta)\hat{\boldsymbol{z}}(t)$$

where $L \in R^{N\times N}$ is the observer gain.

This observer asymptotically converges if we could set the gain L such that all poles of $A(\delta) - LC(\delta)$ have negative real parts for any δ. From the inversion of Eq.(16), the estimate $\hat{\hat{x}}(t)$ is obtained by

$$\hat{\hat{x}}(t) = p[1\ 0\ \cdots\ 0]\hat{\boldsymbol{z}}(t) + m\ .$$

4. Numerical Experiments

We illustrate numerical simulations of the formal linearization and the time-delay nonlinear observer.

4.1. Formal Linearization

Consider the following time-delay nonlinear system :

$$\dot{x}(t) = -x^3(t) + \cos x(t) + 2x(t-\ell)\ , \qquad (20)$$

$$\varphi(\theta) = x_0 \quad (-\ell \le \theta < 0),$$

$$x_0 = 0,\ \ell = 0.2\ .$$

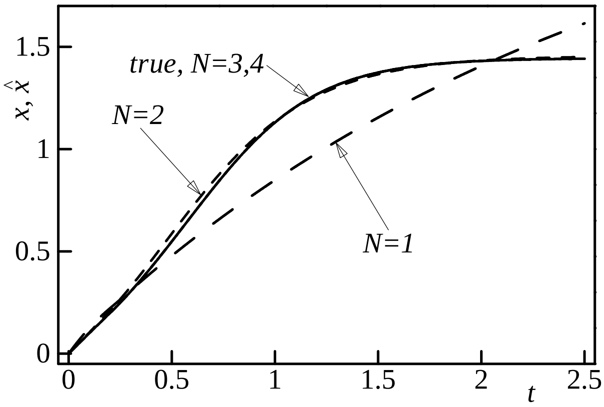

Figure 1: x and $\hat{x}$ by the proposed linearization method

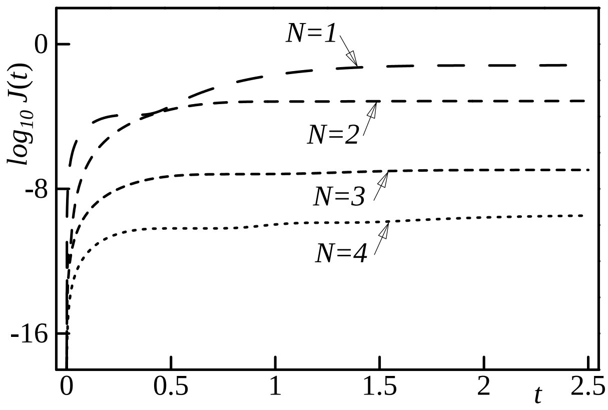

Figure 2: Approximation errors by the linearization

This time-delay nonlinear system (Eq.(20)) is transformed into a linear one (Eq.(15)). In this case,

$$f(x(t)) = -x^3(t) + \cos x(t),$$

$$g\big(x(t-\ell)\big) = 2x(t-\ell) = 2\delta x(t)\,.$$

When the order of the linearization function is $N = 3$, the linearization function (Eq.(9)) is

$$\boldsymbol{\phi}(y) = [y,\ 2y^2 - 1,\ 4y^3 - 3y]^T\ ,$$

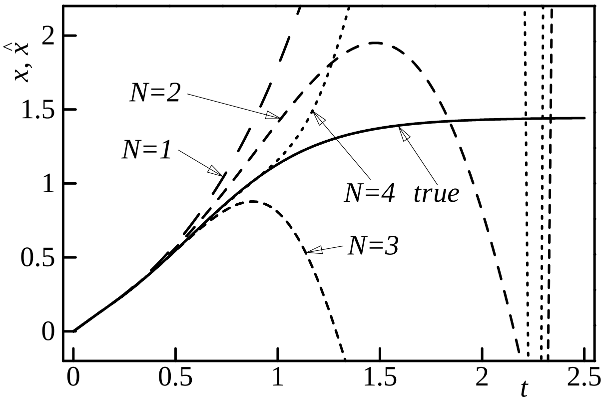

Figure 3: x and $\hat{x}$ by the Taylor expansion method

coefficients of the formal time-delay linear system (Eq.(15)) are

$$A(\delta) = \begin{pmatrix} -2.54 + 2\delta & -0.99 & -0.14 \\ 3.19 & -5.38 + 4\delta & -1.97 \\ -15.7 + 12\delta & 4.79 & -8.06 + 6\delta \end{pmatrix},$$

$$B(\delta) = [1.29,\ -5.09 + 4\delta,\ 0.916]^T\ ,$$

and the initial condition is

$$\boldsymbol{\phi}(\theta) = [y_0,\ 2y_0^2 - 1,\ 4y_0^3 - 3y_0]^T \quad (-\ell \le \theta \le 0)$$

where

$$y_0 = \frac{x_0 - m}{p}$$

in this problem. Solving a formal linear system (Eq.(15)), the approximated solution $\hat{x}(t)$ of the nonlinear system (Eq.(20)) is obtained by the inversion (Eq.(16)).

Fig. 1 shows the true value $x(t)$ which is solution of Eq.(20), and the approximated values $\hat{x}(t)$ when the order of the linearization function is varied as $N = 1$ to 4. Fig. 2 depicts these errors by

$$J(t) = \int_0^t \big(x(\tau) - \hat{x}(\tau)\big)^2 d\tau.$$

For comparison, Fig. 3 shows $x(t)$ and $\hat{x}(t)$ by the conventional method based on Taylor expansion [8] when the order of the linearization function $N = 1$ to 4.

4.2. Time-Delay Nonlinear Observer

Let a time-delay nonlinear system be the same as before in Eq.(20):

$$\dot{x}(t) = -x^3(t) + \cos x(t) + 2x(t-\ell)\ , \qquad (21)$$

$$\varphi(\theta) = x_0 \quad (-\ell \le \theta < 0),$$

$$x_0 = 0,\ \ell = 0.2\ .$$

and a measurement equation be given by

$$\eta(t) = C(\delta)\boldsymbol{\phi}(t), \qquad (22)$$

$$C(\delta) = \begin{pmatrix} \delta & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{pmatrix}.$$

From the given nonlinear system Eqs.(21) and (22), an observer (Eq.(19)) is obtained by the formal linearization in Section 3 as follows:

$$\dot{\hat{\mathbf{z}}}(t) = A(\delta)\hat{\mathbf{z}}(t) + L(\eta - \hat{\eta}) + B(\delta), \qquad (23)$$

$$\hat{\eta} = C(\delta)\hat{\mathbf{z}}(t) .$$

If we set the parameters for the time-delay nonlinear observer as $N = 3$ by

$$L = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 12 & 0 & 6 \end{pmatrix}, \qquad (24)$$

the poles of

$$A(\delta) - LC(\delta) = \begin{pmatrix} -2.54 & -0.99 & -0.14 \\ 3.19 & -5.38 & -1.97 \\ -15.7 & 4.79 & -8.06 \end{pmatrix}$$

are

$$\{-4.29 + 2.43i,\ -4.29 - 2.43i,\ -7.4\} .$$

Thus, an estimate of this observer (Eq.(23)) asymptotically converges to the true value.

Fig. 4 shows the true value $x(t)$ and the estimate $\hat{\hat{x}}(t)$ when the unknown initial value is $\hat{\hat{x}}(0) = 2$ and the order of the linearization function is $N = 3$.

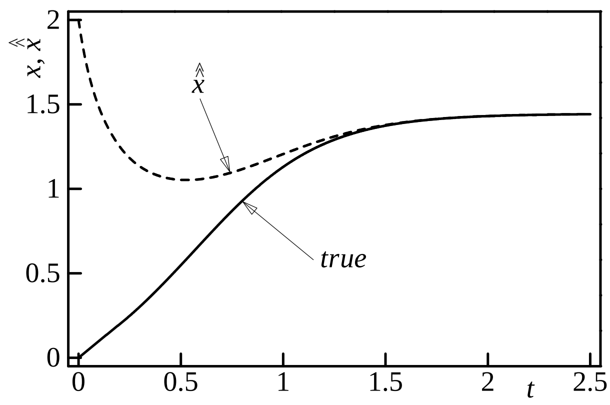

Figure 4: Estimate by time-delay nonlinear observer

5. Conclusions

We have develop a formal linearization method for time-delay nonlinear systems using Chebyshev expansion in order to improve accuracy of the linearization. As an application of this method, we synthesize a time-delay nonlinear observer. Numerical experiments show that accuracy of this method is better than the conventional method based on Taylor expansion and improves as the order of the linearization function increases.

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