

# A Fuzzy Estimation Theory for Available Operation of Extremely Complicated Large-Scale Network Systems

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**Abstract**—In this paper, we shall describe about a fuzzy estimation theory based on the concept of set-valued operators, suitable for available operation of extremely complicated large-scale network systems. Fundamental conditions for availability of system behaviors of such network systems are clarified in a form of  $\beta$ -level fixed point theorem for system of fuzzy-set-valued operators. Here, the proof of this theorem is accomplished in a weak topology introduced into the Banach space.

# 1. Introduction

In order to effectively evaluate, control and maintain extremely complicated large-scale networks, as a whole, the author has recommended to introduce some connectedblock structure: *i.e.*, whole networks might be separated into several blocks which are carefully self-evaluated, selfcontrolled and self-maintained by themselves, and so, which are originally self-sustained systems. However, by always carefully watching each other, whenever they observe and detect that some other block is in ill-condition by some accidents, every block can repair and sustain that ill-conditioned block, through inter-block connections, at once. This style of maintenance of the system is sometimes called as locally autonomous, but the author recommends that only the ultimate responsibility on observation and regulation of whole system might be left for headquarter itself, which is organized over all blocks [1].

Here, let us consider Banach spaces  $X_i$   $(i = 1, \dots, n)$ and  $Y_j$   $(j = 1, \dots, n)$ , and their bounded convex closed subsets  $X_i^{(0)}$  and  $Y_j^{(0)}$ , respectively, corresponding to each block,  $B_i$  and  $B_j$  of whole network system. Let us introduce operators  $f_{ij} : X_i \to Y_j$  such that  $f_{ij}(X_i^{(0)}) \subset Y_j^{(0)}$  and let  $f_{ij}$ be completely continuous on  $X_i^{(0)}$ .

For each block  $B_i(i = 1, \dots, n)$ , dynamics of system behaviors can be represented originally by simple equations:

$$x_i = \alpha_i f_{ii}(x_i), \quad (i = 1, \cdots, n), \tag{1}$$

where  $\alpha_i$  is a continuous operator:  $Y_i^{(0)} \to X_i^{(0)}$ . These equations have solutions  $x_i^*$  in every  $X_i^{(0)}(i = 1, \dots, n)$ , according to the well-known Schauder's type of fixed point theorem. Of course, these solutions represent original values of system behaviors. On the otherhand,  $f_{ij}(j \neq i)$  represents the operation fed-back through all other blocks  $(j \neq i)$  into the original *i*-th block, and  $f_{ji}(j \neq i)$  represents interblock connections from all other blocks, in order to repair and sustain the *i*-th block performance.

However, the fluctuation imposed on the actual system is nondeterministic rather than deterministic. Therefore, it is reasonable to consider some suitable subset of the range of system behavior, in place of single ideal point, as target which the behavior must reach under influence of system control. Now, we can name it as an "available range" of the system behavior. Thus, by the available range, we mean the range of behavior, in which every behavior effectively satisfies good conditions beforehand specified, as a set of ideal behaviors. From such a point of view, the theory for fluctuation imposed on the system should be developed concerning the set-valued operator.

Several years ago, the author gave a general type of fixed point theorem for the system of set-valued operator equations, in order to treat with extremely complicated largescale network systems [1], [2], [3]. Namely, by introducing *n* set-valued operators  $G_i : X_i \times \prod_j^n Y_j \times \prod_j^n Y_i \to \mathcal{F}(X_i)$ (the family of all non-empty closed compact subsets of  $X_i$ ) ( $i = 1, \dots, n$ ), where  $\prod_j^n Y_j$  means the direct product of *n*  $Y_j$ 's, for any  $j \in \{1, \dots, n\}$ , and  $\prod_i^n Y_i$  means direct product of *n*  $Y_i$ 's, for fixed *i*, the author presented important fixed point theorems on the system of set-valued operator equations:

$$x_i \in G_i(x_i; f_{i1}(x_i), \cdots, f_{in}(x_i); f_{1i}(x_1), \cdots, f_{ni}(x_n)),$$
  
(i = 1, \dots, n). (2)

For convenience' sake, let us define a direct product space  $\mathbf{Y}_i \stackrel{\triangle}{=} \prod_j^n Y_j \times \prod^n Y_i$  and also let  $\mathbf{Y}_i^{(0)}$  be a non-empty bounded closed convex subset of  $\mathbf{Y}_i$ . Here, let us consider a vector  $v_i \stackrel{\triangle}{=} (x_i, \dots, x_i; x_1, \dots, x_n) \in V_i$  and an operator  $f_i(v_i) : V_i \to \mathbf{Y}_i$  by

$$f_i(v_i) \stackrel{\scriptscriptstyle \Delta}{=} (f_{i1}(x_i), \cdots, f_{in}(x_i); f_{1i}(x_1), \cdots, f_{ni}(x_n)) .$$
(3)

Here, we know that  $y_{ij} \triangleq f_{ij}(x_i) \in Y_j$ ,  $y_{ji} \triangleq f_{ji}(x_j) \in Y_i$  and  $\mathbf{y}_i \triangleq (y_{i1}, \dots, y_{in}; y_{1i}, \dots, y_{ni}) \in \mathbf{Y}_i$ . Therefore, we have a simple representation of the system of set-valued operator equations (2), as follows:

$$x_i \in G_i(x_i; f_i(v_i)), \ (i = 1, \cdots, n).$$
 (4)

Recently, the author presented a refined estimation theory for such large-scale network systems, by using the fuzzy concept, but under some natural assumptions, at the NOLTA 2004 symposium and further as a paper in the transactions of the IEICE, Fundamentals.[6]

Here, we will approve the same theory with a more refined verification, introducing the weak topology into the Banach space.

## 2. Fuzzy Set and Fuzzy-Set-Valued Operator

First of all, let us consider a family of all fuzzy sets originally introduced by Zadeh [4], in a Banach space *X* with the norm || ||, and let any fuzzy set *A* be characterized by a membership function  $\mu_A(x) : X \to [0, 1]$ . Now, we can consider an  $\alpha$ -level set  $A_\alpha$  of the fuzzy set *A* as  $A_\alpha \stackrel{\Delta}{=} \{\xi \in X | \mu_A(\xi) \ge \alpha\}$ , for any constant  $\alpha \in (0, 1]$ . The fuzzy set *A* is called compact, if all  $\alpha$ -level sets are compact for arbitrary  $\alpha \in (0, 1]$ .

A fuzzy-set-valued operator *G* from *X* into *X* is defined by  $G : X \to \mathcal{F}(X)$ , where  $\mathcal{F}(X)$  is a family of all non-empty , bounded and closed fuzzy sets in *X*. If a point  $x \in X$  is mapped to a fuzzy set G(x), the membership function of G(x) at the point  $\xi \in X$  is represented by  $\mu_{G(x)}(\xi)$ .

For convenience, let us introduce a useful notation: for an arbitrarily specified constant  $\beta \in (0, 1]$ , a point *x* belongs to the  $\beta$ -level set  $A_{\beta}$  of the fuzzy set *A*:  $x \in A_{\beta} \stackrel{\triangle}{=} \{\xi \in X \mid \mu_A(\xi) \ge \beta\}$  is denoted by  $x \in_{\beta} A$  [5].

Here, let us introduce a new concept of  $\beta$ -level fixed point: for the fuzzy set G(x), if there exists a point  $x^*$  such that  $x^* \in_{\beta} G(x^*)$ , then  $x^*$  is called  $\beta$ -level fixed point of the fuzzy-set-valued operator G [5].

Now, let us remember that we have introduced a new metric into the space of fuzzy sets [5, 6].

**Definition 1** Let us consider a Banach space  $X, \rho$ . For any fixed constant  $\beta \in (0, 1]$ , the  $\beta$ -level metric  $\rho_{\beta}$  between a point  $x \in X$  and a fuzzy set A is defined as follows:

$$\rho_{\beta}(x,A) \stackrel{\scriptscriptstyle \Delta}{=} \inf_{\beta \le \alpha \le 1} d_{\alpha}(x,A), \tag{5}$$

where

$$d_{\alpha}(x,A) \stackrel{\scriptscriptstyle \triangle}{=} \begin{cases} \inf_{y \in A_{\alpha}} ||x - y|| & \text{if } \alpha \le \alpha_{A}, \\ \inf_{y \in A_{\alpha_{A}}} ||x - y|| & \text{if } \alpha > \alpha_{A}. \end{cases}$$
(6)

Here,  $\alpha_A \stackrel{\triangleq}{=} \sup_{x \in X} \mu_A(x)$ . And also, for any fixed constant  $\beta \in (0, 1]$ , by means of the Hausdorff metric  $d_H$ , the  $\beta$ -level metric  $\mathcal{H}_\beta$  between two fuzzy sets A and B is introduced as follows:

$$\mathcal{H}_{\beta}(A,B) \stackrel{\scriptscriptstyle \Delta}{=} \sup_{\beta \le \alpha \le 1} D_{\alpha}(A,B), \tag{7}$$

where  $D_{\alpha}$  is defined as

$$D_{\alpha}(A, B) \stackrel{\scriptscriptstyle \triangle}{=} \begin{cases} d_{H}(A_{\alpha}, B_{\alpha}) \\ \text{if } \alpha \leq \min\{\alpha_{A}, \alpha_{B}\}, \\ d_{H}(A_{\alpha_{A}}, B_{\alpha}) \\ \text{if } \alpha_{A} < \alpha \leq \alpha_{B}, \\ d_{H}(A_{\alpha}, B_{\alpha_{B}}) \\ \text{if } \alpha_{A} \geq \alpha > \alpha_{B}, \\ d_{H}(A_{\alpha_{A}}, B_{\alpha_{B}}) \\ \text{if } \alpha > \max\{\alpha_{A}, \alpha_{B}\}. \end{cases}$$
(8)

Here,  $\alpha_B \stackrel{\triangle}{=} \sup_{x \in X} \mu_B(x)$  and the Hausdorff metric  $d_H$  between two sets  $S_1$  and  $S_2$  is defined by

$$d_H(S_1, S_2) \stackrel{\vartriangle}{=} \max\{\sup\{d(x_1, S_2) | x_1 \in S_1\}, \\ \sup\{d(x_2, S_1) | x_2 \in S_2\}\},\$$

where  $d(x, S) \stackrel{\triangle}{=} \inf\{||x - y|| \mid y \in S\}$  is the distance between *a* point *x* and *a* set *S*.

In order to give a new methodology for the discussion more sophisticated than the one by usual set-valued operators, the author presented mathematical theories based on the concept of  $\beta$ -level fixed point, by establishing fixed point theorems for  $\beta$ -level fuzzy-set-valued nonlinear operators which describe detailed characteristics of such fuzzyset-valued nonlinear operator equations, for every level  $\beta \in (0, 1]$  [5, 6].

### 3. System of Fuzzy-Set-Valued Operator Equations

Now, let us introduce a more fine estimation theory for available operation of large-scale system of set-valued operators (4), by introducing  $\beta$ -level fuzzy estimation.

Originally, these sets are crisp. However, in order to introduce more fine estimation into these resultant fluctuation sets, here we can reconsider anew these sets  $G_i$  as fuzzy sets. Then, let us replace the above described crisp sets  $G_i(x_i; f_i(v_i))$  by fuzzy sets with same notations, accompanied with suitable membership functions  $\mu_{Gi}(\xi_i), \xi_i \in X_i$ , which should be properly introduced corresponding to conscious planning for the fine evaluation of resultant fluctuations themselves.

In order to realize a more precise analysis, let us introduce different values of  $\beta$  as  $\beta_i$  (i = 1, ..., n), consciously selected corresponding to every block  $B_i$ .

Now, for any fixed constant  $\beta_i \in (0, 1]$  (i = 1, ..., n), we can introduce a system of  $\beta_i$ -level fuzzy-set-valued nonlinear operator equations:

$$x_i \in_{\beta_i} G_i(x_i; f_i(v_i)), \ (i = 1, \cdots, n).$$
 (9)

If there exists a set of  $\beta_i$ -level fixed points  $\{x_i^*\}$  in  $X_i^{(0)}$ ( $i = 1, \dots, n$ ), which satisfy the system of  $\beta_i$ -level fuzzyset-valued operator equations (9), each  $x_i^*$  can be considered as a  $\beta_i$ -level likelihood behavior of the block  $B_i$ , ( $i = 1, \dots, n$ ). Here, this  $\beta_i$ -level likelihood behavior  $x_i^*$  can be found in a closed domain in which the membership function  $\mu_{G_i(x_i^*;f_i(v_i^*))}(\xi_i)$  has value larger than or equal to  $\beta_i$ .

As shown in the Introduction, we know that the original system of nonlinear operator equations (1) has the desired solution  $x_i^{(0)}$  in every  $X_i^{(0)}$  ( $i = 1, \dots, n$ ). In order to establish a fine estimation in the fuzzy space  $X_i^{(0)}$ , we can conveniently specify the membership function  $\mu_{G_i}(\xi_i)$ in bell-shape such that the desired solution  $x_i^{(0)}$  gives the maximum value of  $\mu_{G_i}$ :  $\beta_{iM} \stackrel{\triangle}{=} \max_{\xi_i \in X_i} \mu_{G_i}(x_i^*;f_i(v_i^*))(\xi_i)$ . As results, if we arbitrarily select two values  $\beta_i, \beta'_i \in (0, 1]$  such that  $\beta_i > \beta'_i$ , we can expect a more fine likelihood fuzzy estimation for  $\beta_i$ -level than the one for  $\beta'_i$ -level.

In the case of single variable system, we can illustrate an example of the  $\beta_i$ -level likelihood estimation for a bellshape membership function, as in Figure 1.



Figure 1: An illustrative example of the  $\beta_i$ -level estimation for bell-shape fuzzy fluctuation from the desired behavior  $x_i^{(0)}$ , in the case of single variable system

On the one hand, when the signal  $x_i^*$  is found in a sufficiently small preassigned closed subset  $\tilde{X}_i^{(0)} \subset X_i^{(0)}$ , containing the desired signal  $x_i^{(0)}$ ,  $x_i^*$  can be considered as "available". Henceforth, let us call such an  $x_i^*$  as an "available" as  $\beta_i$ -level likelyhood behavior.

If we select  $\beta_i \in (0, 1]$  sufficiently high, *i.e.*, near to unity, then the  $\beta_i$ -level set  $G_{i\beta_i} \triangleq \{\xi_i \in X_i | \mu_{G_i}(\xi_i) \ge \beta_i\}$  is so small that  $G_{i\beta_i} \subset \tilde{X}_i^{(0)}$ , and as result, the solution  $x_i^*$  becomes to be available, as a  $\beta_i$ -level likelihood behavior of the block  $B_i$ .

# 4. Fixed Point Theorem For System of β<sub>i</sub>-level Fuzzy-Set-Valued Operators

Here, we will present a mathematical theory of the fixed point theorem for such a general system of  $\beta_i$ -level fuzzy-

set-valued operator equations.

For the first step, let us introduce reflexive, or uniformly convex, real Banach spaces  $X_i$   $(i = 1, \dots, n)$ , in which the norm is represented by  $\|\cdot\|$ , and also their non-empty bounded closed convex subsets  $X_i^{(0)}$   $(i = 1, \dots, n)$ . Let  $X'_i$ be the dual space of  $X_i$  and let us introduce a weak topology  $\sigma(X_i, X'_i)$  into  $X_i$ . Then,  $X_i$  is locally convex topological linear space, and therefore,  $X_i^{(0)}$  is weakly closed and weakly compact. Further, let us consider another real Banach spaces  $Y_j$   $(j = 1, \dots, n)$  in which the norm is represented by  $\|\cdot\|$ .

Now, let us introduce a series of assumptions:

**Assumption 1** Let the operator  $f_{ij}: X_i^{(0)} \to f_{ij}(X_i^{(0)}) \subset Y_j$ be completely continuous in the sense that when a weakly convergent net  $\{x_i^v\}$  ( $v \in J$ : a directive set) weakly converges to  $\bar{x}_i$ , then the sequence  $\{f_{ij}(x_i^v)\}$  has a subsequence which strongly converges to  $f_{ij}(\bar{x}_i)$  in  $Y_j$ .

**Assumption 2** Let the fuzzy-set-valued operator  $G_i$ :  $X_i^{(0)} \times \mathbf{Y}_i \to \mathcal{F}(X_i)$  (a family of all non-empty closed compact subsets of  $X_i$ ) satisfies the following Lipschitz condition with respect to the  $\beta_i$ -level metric  $\mathcal{H}_{\beta_i}$ : that is, there are two kinds of constants  $0 < p_i < 1$  and  $q_i > 0$  such that for any  $x_i^{(1)}, x_i^{(2)} \in X_i$ , for any  $\mathbf{y}_i^{(1)}, \mathbf{y}_i^{(2)} \in \mathbf{Y}_i$ ,  $G_i$ 's satisfy inequalities:

$$\begin{aligned} \mathcal{H}_{\beta_i} \left( G_i(x_i^{(1)}; \mathbf{y}_i^{(1)}), G_i(x_i^{(2)}; \mathbf{y}_i^{(2)}) \right) \\ &\leq p_i \cdot \left\| x_i^{(1)} - x_i^{(2)} \right\| + q_i \cdot \left\| \mathbf{y}_i^{(1)} - \mathbf{y}_i^{(2)} \right\|. 
\end{aligned} \tag{10}$$

where,  $\|\mathbf{y}_i\| \stackrel{\scriptscriptstyle \Delta}{=} \sum_{j=1}^n \|y_{ij}\| + \sum_{j=1}^n \|y_{ji}\|.$ 

**Assumption 3** For any  $x_i \in X_i^{(0)}$  and  $f_i(v_i) \in Y_i$ ,  $G_i^{(0)}(x_i \ f_i(v_i)) \stackrel{\scriptscriptstyle \triangle}{=} G_i(x_i; \ f_i(v_i)) \cap X_i^{(0)} \neq \phi$ . Moreover, there exist projection points  $\tilde{x}_i' \in X_i^{(0)}$  of arbitrary point  $x_i' \in X_i^{(0)}$ upon the set  $G_{i\beta_i}^{(0)}(x_i \ f_i(v_i))$  such that

$$\|\tilde{x}_{i}' - x_{i}'\| = \min\left\{ \|x_{i}' - z_{i}\| \mid z_{i} \in G_{i\beta_{i}}^{(0)}(x_{i} \ f_{i}(v_{i})) \right\}, \quad (11)$$

where  $G_{i\beta_i} \stackrel{\scriptscriptstyle \Delta}{=} \{\xi \in X_i | \mu_{G_i}(\xi) \ge \beta_i\}.$ 

**Assumption 4 (Rockafellar [7])** For any  $x_i^{(1)}, x_i^{(2)} \in X_i^{(0)}$ , and for any constant r(0 < r < 1), uniformly with respect to every  $\mathbf{y}_i \in \mathbf{Y}_i$ ,  $G_i$  satisfies the relation:

$$r \cdot G_{i\beta_{i}}\left(x_{i}^{(1)}; \mathbf{y}_{i}\right) + (1-r) \cdot G_{i\beta_{i}}\left(x_{i}^{(2)}; \mathbf{y}_{i}\right) \\ \subset G_{i\beta_{i}}\left(r \cdot x_{i}^{(1)} + (1-r) \cdot x_{i}^{(2)}; \mathbf{y}_{i}\right) .$$

$$(12)$$

Then, we have an important lemma:

**Lemma 1** For all i (i = 1, ..., n), let us adopt arbitrary points  $x_i \equiv z_i^{(0)} \in X_i^{(0)}$  and also fix all values of  $f_i(v_i^{(0)}), v_i^{(0)} \triangleq (x_i^{(0)}, ..., x_i^{(0)}; x_1^{(0)} \cdots x_n^{(0)})$ . Now, for every i, let us introduce a sequence  $\{z_i^k\}$  (k = 0, 1, 2, ...), starting from the above-adopted point  $z_i^{(0)}$ , and with each

 $z_i^k \in X_i^{(0)}$  as a projection point of  $z_i^{k-1} \in X_i^{(0)}$  upon the set  $G_{i\beta_i}(z_i^{k-1}; f_i(v_i^{(0)}))$ . Then, this sequence  $\{z_i^k\}$  ( $k = 0, 1, 2, \cdots$ ) is a Cauchy sequence, having its own limit points  $\bar{z}_i \in X_i^{(0)}$ , such that

$$\bar{z}_i \in_{\beta_i} G_i^{(0)}\left(\bar{z}_i; f_i(v_i^{(0)})\right), \ (i = 1, \cdots, n) \ . \tag{13}$$

Here, the correspondence from the starting point  $z_i^{(0)} \equiv x_i \in X_i^{(0)}$  to the limit points  $\overline{z}_i \in X_i^{(0)}$  is multivalued, in general, and hence, by this correspondence we can define a set-valued operator  $H_i : x_i \to {\overline{z}_i}$  in  $X_i$ ; *i.e.*,  $\overline{z}_i \in H_i(x_i)$ . If this set-valued operator  $H_i$  has a fixed point  $x_i^*$ : *i.e.*,  $x_i^* \in H_i(x_i^*)$ , then it satisfies the system of equations:

$$x_i^* \in_{\beta_i} G_i^{(0)} \left( x_i^*; f_i(v_i^*) \right), \ (i = 1, \cdots, n)$$
(14)

by Eq. (13) with the corresponding  $v_i^* \stackrel{\triangle}{=} (x_i^*, \dots, x_i^*; x_1^*, \dots, x_n^*).$ 

This  $x_i^*$  is to be the solution of the system of  $\beta_i$ -level fuzzy-set-valued operator equations (14), refined in the weak topology from Eq. (4).

Here, we can easily recognize that the operator  $H_i$  is upper semicontinuous and the range of  $H_i$  is closed and convex.

Therefore, in order to verify the existence of the fixed point  $x_i^*$  of  $H_i$ , now, we can apply the well-known fixed point theorem for set-valued operator:

**Lemma 2 (Ky Fan [8])** Let  $X_i$  be a locally convex topological linear space, and  $X_i^{(0)}$  be a non-empty convex compact subset of  $X_i$ . Let  $\mathcal{H}_c(X_i^{(0)})$  be the family of all nonempty closed convex subset of  $X_i^{(0)}$ . Then, for any upper semicontinuous set-valued operator  $H_i$ :  $X_i^{(0)} \to \mathcal{H}_c(X_i^{(0)})$ , there exists a fixed point  $x_i^* \in X_i^{(0)}$  such that  $x_i^* \in H_i(x_i^*)$ .

As a result, we have a theorem:

**Theorem 1** Let  $X_i$  be a reflexive, or uniformly convex, real Banach space, and  $X_i^{(0)}$  be a non-empty bounded closed convex subset of  $X_i$ . By the dual space  $X'_i$ , let us introduce a weak topology  $\sigma(X_i, X'_i)$  into  $X_i$ . Let  $f_{ij}$  and  $G_i$  be deterministic and fuzzy-set-valued operators, respectively, which satisfy the series of assumptions 1 to 4. Then, we have a Cauchy sequence  $\{z_i^k\} \subset X_i^{(0)}$  ( $k = 0, 1, 2, \cdots$ ), introduced by the succesive procedure in Lemma 1. This sequence has a set of limit points  $\{\bar{z}_i\}$ , and we can define a set-valued operator  $H_i$  by the correspondence from the arbitrary starting point  $z_i^{(0)} \equiv x_i \in X_i^{(0)}$  to the set of limit points  $\{\bar{z}_i\}$  in  $X_i^{(0)}$ :  $\bar{z}_i \in H_i(x_i)$ . This set-valued operator  $H_i$ has a fixed point  $x^*$  in  $X_i^{(0)}$ , which is, in turn, the solution of the system of  $\beta_i$ -level fuzzy-set-valued operator equations (14).

# 5. Concluding Remark

Thus, the fluctuation analysis of this type of large-scale network systems, undergone by undesirable uncertain fluctuations, can be successfully accomplished at arbitrarily fine-level of estimation, by immediate application of the here-presented fixed point theorem for system of  $\beta_i$ -level fuzzy-set-valued nolinear operators, with consciously selected different value of parameter  $\beta_i$ , for every block  $B_i$ .

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