

A Fuzzy Estimation Theory for Available Operation of Extremely Complicated Large-Scale Network Systems

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Abstract—In this paper, we shall describe about a fuzzy estimation theory based on the concept of set-valued operators, suitable for available operation of extremely complicated large-scale network systems. Fundamental conditions for availability of system behaviors of such network systems are clarified in a form of β -level fixed point theorem for system of fuzzy-set-valued operators. Here, the proof of this theorem is accomplished in a weak topology introduced into the Banach space.

1. Introduction

In order to effectively evaluate, control and maintain extremely complicated large-scale networks, as a whole, the author has recommended to introduce some connected-block structure: *i.e.*, whole networks might be separated into several blocks which are carefully self-evaluated, self-controlled and self-maintained by themselves, and so, which are originally self-sustained systems. However, by always carefully watching each other, whenever they observe and detect that some other block is in ill-condition by some accidents, every block can repair and sustain that ill-conditioned block, through inter-block connections, at once. This style of maintenance of the system is sometimes called as locally autonomous, but the author recommends that only the ultimate responsibility on observation and regulation of whole system might be left for headquarter itself, which is organized over all blocks [1].

Here, let us consider Banach spaces X_i ($i = 1, \dots, n$) and Y_j ($j = 1, \dots, n$), and their bounded convex closed subsets $X_i^{(0)}$ and $Y_j^{(0)}$, respectively, corresponding to each block, B_i and B_j of whole network system. Let us introduce operators $f_{ij} : X_i \rightarrow Y_j$ such that $f_{ij}(X_i^{(0)}) \subset Y_j^{(0)}$ and let f_{ij} be completely continuous on $X_i^{(0)}$.

For each block B_i ($i = 1, \dots, n$), dynamics of system behaviors can be represented originally by simple equations:

$$x_i = \alpha_i f_{ii}(x_i), \quad (i = 1, \dots, n), \quad (1)$$

where α_i is a continuous operator: $Y_i^{(0)} \rightarrow X_i^{(0)}$. These equations have solutions x_i^* in every $X_i^{(0)}$ ($i = 1, \dots, n$), according to the well-known Schauder's type of fixed point theorem. Of course, these solutions represent original values of system behaviors. On the otherhand, f_{ij} ($j \neq i$) represents the operation fed-back through all other blocks ($j \neq i$)

into the original i -th block, and f_{ji} ($j \neq i$) represents inter-block connections from all other blocks, in order to repair and sustain the i -th block performance.

However, the fluctuation imposed on the actual system is nondeterministic rather than deterministic. Therefore, it is reasonable to consider some suitable subset of the range of system behavior, in place of single ideal point, as target which the behavior must reach under influence of system control. Now, we can name it as an "available range" of the system behavior. Thus, by the available range, we mean the range of behavior, in which every behavior effectively satisfies good conditions beforehand specified, as a set of ideal behaviors. From such a point of view, the theory for fluctuation imposed on the system should be developed concerning the set-valued operator.

Several years ago, the author gave a general type of fixed point theorem for the system of set-valued operator equations, in order to treat with extremely complicated large-scale network systems [1], [2], [3]. Namely, by introducing n set-valued operators $G_i : X_i \times \prod_{j \neq i}^n Y_j \times \prod_{i=1}^n Y_i \rightarrow \mathcal{F}(X_i)$ (the family of all non-empty closed compact subsets of X_i) ($i = 1, \dots, n$), where $\prod_{j \neq i}^n Y_j$ means the direct product of n Y_j 's, for any $j \in \{1, \dots, n\}$, and $\prod_{i=1}^n Y_i$ means direct product of n Y_i 's, for fixed i , the author presented important fixed point theorems on the system of set-valued operator equations:

$$x_i \in G_i(x_i; f_{i1}(x_i), \dots, f_{in}(x_i); f_{1i}(x_1), \dots, f_{ni}(x_n)), \quad (2)$$

$$(i = 1, \dots, n).$$

For convenience' sake, let us define a direct product space $\mathbf{Y}_i \triangleq \prod_{j \neq i}^n Y_j \times \prod_{i=1}^n Y_i$ and also let $\mathbf{Y}_i^{(0)}$ be a non-empty bounded closed convex subset of \mathbf{Y}_i . Here, let us consider a vector $v_i \triangleq (x_i, \dots, x_i; x_1, \dots, x_n) \in V_i$ and an operator $f_i(v_i) : V_i \rightarrow \mathbf{Y}_i$ by

$$f_i(v_i) \triangleq (f_{i1}(x_i), \dots, f_{in}(x_i); f_{1i}(x_1), \dots, f_{ni}(x_n)). \quad (3)$$

Here, we know that $y_{ij} \triangleq f_{ij}(x_i) \in Y_j$, $y_{ji} \triangleq f_{ji}(x_j) \in Y_i$ and $\mathbf{y}_i \triangleq (y_{i1}, \dots, y_{in}; y_{1i}, \dots, y_{ni}) \in \mathbf{Y}_i$. Therefore, we have a simple representation of the system of set-valued operator equations (2), as follows:

$$x_i \in G_i(x_i; f_i(v_i)), \quad (i = 1, \dots, n). \quad (4)$$

Recently, the author presented a refined estimation theory for such large-scale network systems, by using the fuzzy concept, but under some natural assumptions, at the NOLTA 2004 symposium and further as a paper in the transactions of the IEICE, Fundamentals.[6]

Here, we will approve the same theory with a more refined verification, introducing the weak topology into the Banach space.

2. Fuzzy Set and Fuzzy-Set-Valued Operator

First of all, let us consider a family of all fuzzy sets originally introduced by Zadeh [4], in a Banach space X with the norm $\| \cdot \|$, and let any fuzzy set A be characterized by a membership function $\mu_A(x) : X \rightarrow [0, 1]$. Now, we can consider an α -level set A_α of the fuzzy set A as $A_\alpha \triangleq \{\xi \in X | \mu_A(\xi) \geq \alpha\}$, for any constant $\alpha \in (0, 1]$. The fuzzy set A is called compact, if all α -level sets are compact for arbitrary $\alpha \in (0, 1]$.

A fuzzy-set-valued operator G from X into X is defined by $G : X \rightarrow \mathcal{F}(X)$, where $\mathcal{F}(X)$ is a family of all non-empty, bounded and closed fuzzy sets in X . If a point $x \in X$ is mapped to a fuzzy set $G(x)$, the membership function of $G(x)$ at the point $\xi \in X$ is represented by $\mu_{G(x)}(\xi)$.

For convenience, let us introduce a useful notation: for an arbitrarily specified constant $\beta \in (0, 1]$, a point x belongs to the β -level set A_β of the fuzzy set A : $x \in A_\beta \triangleq \{\xi \in X | \mu_A(\xi) \geq \beta\}$ is denoted by $x \in_\beta A$ [5].

Here, let us introduce a new concept of β -level fixed point: for the fuzzy set $G(x)$, if there exists a point x^* such that $x^* \in_\beta G(x^*)$, then x^* is called β -level fixed point of the fuzzy-set-valued operator G [5].

Now, let us remember that we have introduced a new metric into the space of fuzzy sets [5, 6].

Definition 1 Let us consider a Banach space X, ρ . For any fixed constant $\beta \in (0, 1]$, the β -level metric ρ_β between a point $x \in X$ and a fuzzy set A is defined as follows:

$$\rho_\beta(x, A) \triangleq \inf_{\beta \leq \alpha \leq 1} d_\alpha(x, A), \quad (5)$$

where

$$d_\alpha(x, A) \triangleq \begin{cases} \inf_{y \in A_\alpha} \|x - y\| & \text{if } \alpha \leq \alpha_A, \\ \inf_{y \in A_{\alpha_A}} \|x - y\| & \text{if } \alpha > \alpha_A. \end{cases} \quad (6)$$

Here, $\alpha_A \triangleq \sup_{x \in X} \mu_A(x)$. And also, for any fixed constant $\beta \in (0, 1]$, by means of the Hausdorff metric d_H , the β -level metric \mathcal{H}_β between two fuzzy sets A and B is introduced as follows:

$$\mathcal{H}_\beta(A, B) \triangleq \sup_{\beta \leq \alpha \leq 1} D_\alpha(A, B), \quad (7)$$

where D_α is defined as

$$D_\alpha(A, B) \triangleq \begin{cases} d_H(A_\alpha, B_\alpha) & \text{if } \alpha \leq \min\{\alpha_A, \alpha_B\}, \\ d_H(A_{\alpha_A}, B_\alpha) & \text{if } \alpha_A < \alpha \leq \alpha_B, \\ d_H(A_\alpha, B_{\alpha_B}) & \text{if } \alpha_A \geq \alpha > \alpha_B, \\ d_H(A_{\alpha_A}, B_{\alpha_B}) & \text{if } \alpha > \max\{\alpha_A, \alpha_B\}. \end{cases} \quad (8)$$

Here, $\alpha_B \triangleq \sup_{x \in X} \mu_B(x)$ and the Hausdorff metric d_H between two sets S_1 and S_2 is defined by

$$d_H(S_1, S_2) \triangleq \max\{\sup\{d(x_1, S_2) | x_1 \in S_1\}, \sup\{d(x_2, S_1) | x_2 \in S_2\}\},$$

where $d(x, S) \triangleq \inf\{\|x - y\| | y \in S\}$ is the distance between a point x and a set S .

In order to give a new methodology for the discussion more sophisticated than the one by usual set-valued operators, the author presented mathematical theories based on the concept of β -level fixed point, by establishing fixed point theorems for β -level fuzzy-set-valued nonlinear operators which describe detailed characteristics of such fuzzy-set-valued nonlinear operator equations, for every level $\beta \in (0, 1]$ [5, 6].

3. System of Fuzzy-Set-Valued Operator Equations

Now, let us introduce a more fine estimation theory for available operation of large-scale system of set-valued operators (4), by introducing β -level fuzzy estimation.

Originally, these sets are crisp. However, in order to introduce more fine estimation into these resultant fluctuation sets, here we can reconsider anew these sets G_i as fuzzy sets. Then, let us replace the above described crisp sets $G_i(x_i; f_i(v_i))$ by fuzzy sets with same notations, accompanied with suitable membership functions $\mu_{G_i}(\xi_i)$, $\xi_i \in X_i$, which should be properly introduced corresponding to conscious planning for the fine evaluation of resultant fluctuations themselves.

In order to realize a more precise analysis, let us introduce different values of β as β_i ($i = 1, \dots, n$), consciously selected corresponding to every block B_i .

Now, for any fixed constant $\beta_i \in (0, 1]$ ($i = 1, \dots, n$), we can introduce a system of β_i -level fuzzy-set-valued nonlinear operator equations:

$$x_i \in_{\beta_i} G_i(x_i; f_i(v_i)), \quad (i = 1, \dots, n). \quad (9)$$

If there exists a set of β_i -level fixed points $\{x_i^*\}$ in $X_i^{(0)}$ ($i = 1, \dots, n$), which satisfy the system of β_i -level fuzzy-set-valued operator equations (9), each x_i^* can be considered as a β_i -level likelihood behavior of the block B_i , ($i = 1, \dots, n$). Here, this β_i -level likelihood behavior x_i^*

can be found in a closed domain in which the membership function $\mu_{G_i(x_i^*; f_i(v_i^*))}(\xi_i)$ has value larger than or equal to β_i .

As shown in the Introduction, we know that the original system of nonlinear operator equations (1) has the desired solution $x_i^{(0)}$ in every $X_i^{(0)}$ ($i = 1, \dots, n$). In order to establish a fine estimation in the fuzzy space $X_i^{(0)}$, we can conveniently specify the membership function $\mu_{G_i}(\xi_i)$ in bell-shape such that the desired solution $x_i^{(0)}$ gives the maximum value of μ_{G_i} : $\beta_{iM} \triangleq \max_{\xi_i \in X_i} \mu_{G_i(x_i^*; f_i(v_i^*))}(\xi_i)$. As results, if we arbitrarily select two values $\beta_i, \beta'_i \in (0, 1]$ such that $\beta_i > \beta'_i$, we can expect a more fine likelihood fuzzy estimation for β_i -level than the one for β'_i -level.

In the case of single variable system, we can illustrate an example of the β_i -level likelihood estimation for a bell-shape membership function, as in Figure 1.

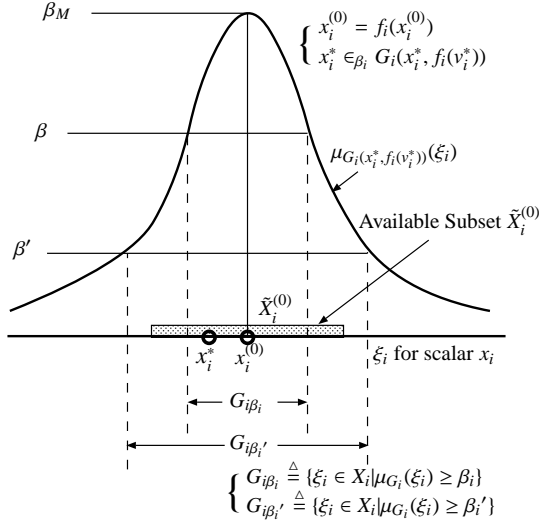


Figure 1: An illustrative example of the β_i -level estimation for bell-shape fuzzy fluctuation from the desired behavior $x_i^{(0)}$, in the case of single variable system

On the one hand, when the signal x_i^* is found in a sufficiently small preassigned closed subset $\tilde{X}_i^{(0)} \subset X_i^{(0)}$, containing the desired signal $x_i^{(0)}$, x_i^* can be considered as “available”. Henceforth, let us call such an x_i^* as an “available” as β_i -level likelihood behavior.

If we select $\beta_i \in (0, 1]$ sufficiently high, i.e., near to unity, then the β_i -level set $G_{i\beta_i} \triangleq \{\xi_i \in X_i | \mu_{G_i}(\xi_i) \geq \beta_i\}$ is so small that $G_{i\beta_i} \subset \tilde{X}_i^{(0)}$, and as result, the solution x_i^* becomes to be available, as a β_i -level likelihood behavior of the block B_i .

4. Fixed Point Theorem For System of β_i -level Fuzzy-Set-Valued Operators

Here, we will present a mathematical theory of the fixed point theorem for such a general system of β_i -level fuzzy-

set-valued operator equations.

For the first step, let us introduce reflexive, or uniformly convex, real Banach spaces X_i ($i = 1, \dots, n$), in which the norm is represented by $\|\cdot\|$, and also their non-empty bounded closed convex subsets $X_i^{(0)}$ ($i = 1, \dots, n$). Let X'_i be the dual space of X_i and let us introduce a weak topology $\sigma(X_i, X'_i)$ into X_i . Then, X_i is locally convex topological linear space, and therefore, $X_i^{(0)}$ is weakly closed and weakly compact. Further, let us consider another real Banach spaces Y_j ($j = 1, \dots, n$) in which the norm is represented by $\|\cdot\|$.

Now, let us introduce a series of assumptions:

Assumption 1 Let the operator $f_{ij} : X_i^{(0)} \rightarrow f_{ij}(X_i^{(0)}) \subset Y_j$ be completely continuous in the sense that when a weakly convergent net $\{x_i^v\}$ ($v \in J$: a directive set) weakly converges to \bar{x}_i , then the sequence $\{f_{ij}(x_i^v)\}$ has a subsequence which strongly converges to $f_{ij}(\bar{x}_i)$ in Y_j .

Assumption 2 Let the fuzzy-set-valued operator $G_i : X_i^{(0)} \times Y_i \rightarrow \mathcal{F}(X_i)$ (a family of all non-empty closed compact subsets of X_i) satisfies the following Lipschitz condition with respect to the β_i -level metric \mathcal{H}_{β_i} : that is, there are two kinds of constants $0 < p_i < 1$ and $q_i > 0$ such that for any $x_i^{(1)}, x_i^{(2)} \in X_i$, for any $y_i^{(1)}, y_i^{(2)} \in Y_i$, G_i 's satisfy inequalities:

$$\mathcal{H}_{\beta_i}(G_i(x_i^{(1)}; y_i^{(1)}), G_i(x_i^{(2)}; y_i^{(2)})) \leq p_i \cdot \|x_i^{(1)} - x_i^{(2)}\| + q_i \cdot \|y_i^{(1)} - y_i^{(2)}\|. \quad (10)$$

where, $\|y_i\| \triangleq \sum_{j=1}^n \|y_{ij}\| + \sum_{j=1}^n \|y_{ji}\|$.

Assumption 3 For any $x_i \in X_i^{(0)}$ and $f_i(v_i) \in Y_i$, $G_i^{(0)}(x_i; f_i(v_i)) \triangleq G_i(x_i; f_i(v_i)) \cap X_i^{(0)} \neq \emptyset$. Moreover, there exist projection points $\tilde{x}_i' \in X_i^{(0)}$ of arbitrary point $x_i' \in X_i^{(0)}$ upon the set $G_{i\beta_i}^{(0)}(x_i; f_i(v_i))$ such that

$$\|\tilde{x}_i' - x_i'\| = \min \{ \|x_i' - z_i\| \mid z_i \in G_{i\beta_i}^{(0)}(x_i; f_i(v_i)) \}, \quad (11)$$

where $G_{i\beta_i} \triangleq \{\xi \in X_i | \mu_{G_i}(\xi) \geq \beta_i\}$.

Assumption 4 (Rockafellar [7]) For any $x_i^{(1)}, x_i^{(2)} \in X_i^{(0)}$, and for any constant $r(0 < r < 1)$, uniformly with respect to every $y_i \in Y_i$, G_i satisfies the relation:

$$r \cdot G_{i\beta_i}(x_i^{(1)}; y_i) + (1-r) \cdot G_{i\beta_i}(x_i^{(2)}; y_i) \subset G_{i\beta_i}(r \cdot x_i^{(1)} + (1-r) \cdot x_i^{(2)}; y_i). \quad (12)$$

Then, we have an important lemma:

Lemma 1 For all i ($i = 1, \dots, n$), let us adopt arbitrary points $x_i \equiv z_i^{(0)} \in X_i^{(0)}$ and also fix all values of $f_i(v_i^{(0)})$, $v_i^{(0)} \triangleq (x_i^{(0)}, \dots, x_i^{(0)}; x_1^{(0)}, \dots, x_n^{(0)})$. Now, for every i , let us introduce a sequence $\{z_i^k\}$ ($k = 0, 1, 2, \dots$), starting from the above-adopted point $z_i^{(0)}$, and with each

$z_i^k \in X_i^{(0)}$ as a projection point of $z_i^{k-1} \in X_i^{(0)}$ upon the set $G_{i\beta_i}(z_i^{k-1}; f_i(v_i^{(0)}))$. Then, this sequence $\{z_i^k\}$ ($k = 0, 1, 2, \dots$) is a Cauchy sequence, having its own limit points $\bar{z}_i \in X_i^{(0)}$, such that

$$\bar{z}_i \in_{\beta_i} G_i^{(0)}(\bar{z}_i; f_i(v_i^{(0)})), \quad (i = 1, \dots, n). \quad (13)$$

Here, the correspondence from the starting point $z_i^{(0)} \equiv x_i \in X_i^{(0)}$ to the limit points $\bar{z}_i \in X_i^{(0)}$ is multivalued, in general, and hence, by this correspondence we can define a set-valued operator $H_i : x_i \rightarrow \{\bar{z}_i\}$ in X_i ; i.e., $\bar{z}_i \in H_i(x_i)$. If this set-valued operator H_i has a fixed point x_i^* : i.e., $x_i^* \in H_i(x_i^*)$, then it satisfies the system of equations:

$$x_i^* \in_{\beta_i} G_i^{(0)}(x_i^*; f_i(v_i^*)), \quad (i = 1, \dots, n) \quad (14)$$

by Eq. (13) with the corresponding $v_i^* \triangleq (x_1^*, \dots, x_i^*, x_{i+1}^*, \dots, x_n^*)$.

This x_i^* is to be the solution of the system of β_i -level fuzzy-set-valued operator equations (14), refined in the weak topology from Eq. (4).

Here, we can easily recognize that the operator H_i is upper semicontinuous and the range of H_i is closed and convex.

Therefore, in order to verify the existence of the fixed point x_i^* of H_i , now, we can apply the well-known fixed point theorem for set-valued operator:

Lemma 2 (Ky Fan [8]) *Let X_i be a locally convex topological linear space, and $X_i^{(0)}$ be a non-empty convex compact subset of X_i . Let $\mathcal{H}_c(X_i^{(0)})$ be the family of all non-empty closed convex subset of $X_i^{(0)}$. Then, for any upper semicontinuous set-valued operator $H_i: X_i^{(0)} \rightarrow \mathcal{H}_c(X_i^{(0)})$, there exists a fixed point $x_i^* \in X_i^{(0)}$ such that $x_i^* \in H_i(x_i^*)$.*

As a result, we have a theorem:

Theorem 1 *Let X_i be a reflexive, or uniformly convex, real Banach space, and $X_i^{(0)}$ be a non-empty bounded closed convex subset of X_i . By the dual space X_i' , let us introduce a weak topology $\sigma(X_i, X_i')$ into X_i . Let f_{ij} and G_i be deterministic and fuzzy-set-valued operators, respectively, which satisfy the series of assumptions 1 to 4. Then, we have a Cauchy sequence $\{z_i^k\} \subset X_i^{(0)}$ ($k = 0, 1, 2, \dots$), introduced by the successive procedure in Lemma 1. This sequence has a set of limit points $\{\bar{z}_i\}$, and we can define a set-valued operator H_i by the correspondence from the arbitrary starting point $z_i^{(0)} \equiv x_i \in X_i^{(0)}$ to the set of limit points $\{\bar{z}_i\}$ in $X_i^{(0)}$: $\bar{z}_i \in H_i(x_i)$. This set-valued operator H_i has a fixed point x^* in $X_i^{(0)}$, which is, in turn, the solution of the system of β_i -level fuzzy-set-valued operator equations (14).*

5. Concluding Remark

Thus, the fluctuation analysis of this type of large-scale network systems, undergone by undesirable uncertain fluctuations, can be successfully accomplished at arbitrarily fine-level of estimation, by immediate application of the here-presented fixed point theorem for system of β_i -level fuzzy-set-valued nonlinear operators, with consciously selected different value of parameter β_i , for every block B_i .

Acknowledgement

The new metric described in Definition 1 is one of results in collaboration with an old student: Assistant Professor Y. Endo, Tsukuba University [5]. Finally, the author is grateful to express his gratitude to Professor S. Oishi, Waseda University, for his kind advice and support to this series of works, and further to Dr. K. Maruyama, Visiting Lecturer, Waseda University, for the L^AT_EX-description of manuscript including the figure, with kind advice.

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