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Stability analysis of amplitude death induced by a partial time-varying delay connection

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Abstract—In this study, we analyze the stability of amplitude death induced by a partial time-varying delay connection. This connection consists of time-invariant delay connections and time-varying delay connections. A linear stability analysis reveals that a partial time-varying delay connection is useful for death induction.

1. Introduction

There has been a growing interest in nonlinear phenomena in coupled oscillators [1]. Amplitude death, which is a diffusive-coupling induced stabilization of unstable steady states in coupled oscillators, has been studied for almost two decades [2]. It was known that amplitude death is not induced in coupled identical oscillators [3]. However, Reddy *el al.* shows that amplitude death can be induced in coupled identical oscillators if the coupling contains a time delay [4]. The delay-induced death has attracted a great attention in nonlinear science and has been intensively studied not only analytically but also experimentally [5–12].

Amplitude death has a potential to suppress an unnecessary self-excited oscillation in engineering systems [6, 7]. However, if the time delay in coupling is long compared with the oscillators period, death cannot occur [4]. Thus, it is impossible to induce amplitude death in the following situations: oscillators apart from each other; oscillators have high frequency. A distributed delay connection [8] and a multiple delay connection [9] are proposed to overcome the above problem. Unfortunately, they have disadvantages: the former is difficult to be implemented in real systems and the latter takes high cost.

Recently, Konishi *et al.* proposed a time-varying delay connection for a pair of limit cycle oscillators [10]. Since the time-varying delay connection would be easy to be implemented and would not be costly, this connection must be one of the strong candidates for overcoming the disadvantages. There has been an experimental verification of the time-varying delay connection in an electric circuit [13]. Furthermore, we have shown that this connection can be applied to network oscillators [14]. However, for the network oscillators, all the time delays have to be varied with high frequency. We can easily imagine that this is a difficult task for a large-scale network consisting of a huge number of oscillators.

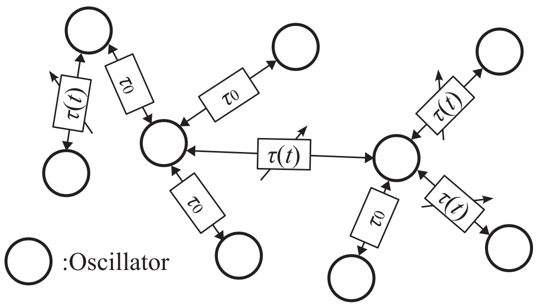


Figure 1: Sketch of network oscillators coupled by the partial time-varying delay connection

The present paper analyzes the stability of amplitude death induced by a partial time-varying delay connection. This connection consists of time-invariant delay connections and time-varying delay connections. It is obviously easier to implement the partial time-varying delay connection than the conventional time-varying delay connection. A linear stability analysis reveals an effectiveness of the partial time-varying delay connection.

2. Network oscillators

Let us consider N identical oscillators (see Fig. 1),

$$\dot{Z}_j(t) = \left\{\mu + i\omega - |Z_j(t)|^2\right\} Z_j(t) + U_j(t), \ (j = 1, \ldots, N), \quad (1)$$

where $Z_j(t) \in \mathbb{C}$ is the state variable of the j-th oscillator. The parameters μ and ω represent the instability of the fixed point and the frequency of the oscillator, respectively. Here, i is denoted as $i := \sqrt{-1}$. The coupling signal $U_j(t) \in \mathbb{C}$ is given by

$$U_j(t) = k\left\{\frac{1}{d_j}\left(\sum_{l=1}^{N}\left(v_{jl}Z_{l,\tau(t)} + w_{jl}Z_{l,\tau_0}\right)\right) - Z_j(t)\right\}, \quad (2)$$

where $Z_{l,\tau(t)} := Z_l(t - \tau(t))$ and $Z_{l,\tau_0} := Z_l(t - \tau_0)$ are the time-varying delayed variable and the time-invariant delayed variable, respectively. v_{jl} and w_{jl} govern the network topology as follows: if the j-th oscillator is connected to the l-th oscillator by a time-varying (time-invariant) delay connection, then $v_{jl} = v_{lj} = 1$ ($w_{jl} = w_{lj} = 1$), otherwise

$v_{jl} = v_{lj} = 0$ ($w_{jl} = w_{lj} = 0$). The degree of the j-th oscillator, that is the number of oscillators connected to the j-th oscillator, is denoted by $d_j := \sum_{l=1}^{N}(v_{jl} + w_{jl})$. The time-varying delay $\tau(t)$ varies around a nominal delay time $\tau_0 > 0$ with amplitude $\delta > 0$,

$$\tau(t) := \tau_0 + \delta f(\Omega t),$$

where Ω is the frequency of variation. $f(x)$ is the periodic sawtooth function,

$$f(x) := \begin{cases} +\frac{2}{\pi}\left(x - \frac{\pi}{2} - 2m\pi\right) & \text{if } x \in [2m\pi, (2m+1)\pi) \\ -\frac{2}{\pi}\left(x - \frac{3\pi}{2} - 2m\pi\right) & \text{if } x \in [(2m+1)\pi, 2(m+1)\pi) \end{cases}, \quad m = 0, 1, 2, \dots .$$

Oscillators (1) with connection (2) have the homogeneous steady state,

$$[Z_1^*, Z_2^*, \dots, Z_N^*]^{\mathrm{T}} = [0, 0, \dots, 0]^{\mathrm{T}}. \quad (3)$$

The dynamics of oscillators (1) with connection (2) around steady state (3) is given by

$$\dot{z}_j(t) = (\mu + i\omega)z_j(t) + u_j(t), \quad (4)$$

where $z_j(t) := Z_j(t) - Z_j^*$ is the variation from steady state (3) and the variation of the coupling signal is given by

$$u_j(t) := k\left\{\frac{1}{d_j}\left(\sum_{l=1}^{N}\left(v_{jl}z_{l,\tau(t)} + w_{jl}z_{l,\tau_0}\right)\right) - z_j(t)\right\}.$$

Equation (4) can be written as

$$\dot{\boldsymbol{X}}(t) = (\boldsymbol{I}_N \otimes \boldsymbol{A}_s)\,\boldsymbol{X}(t) + k(\boldsymbol{D}^{-1} \otimes \boldsymbol{I}_2)\{(\boldsymbol{V} \otimes \boldsymbol{I}_2)\boldsymbol{X}_{\tau(t)} + (\boldsymbol{W} \otimes \boldsymbol{I}_2)\boldsymbol{X}_{\tau_0}\}, \quad (5)$$

where

$$\boldsymbol{X}(t) := \left[\mathrm{Re}[z_1(t)], \mathrm{Im}[z_1(t)], \cdots, \mathrm{Re}[z_N(t)], \mathrm{Im}[z_N(t)]\right]^{\mathrm{T}},$$

$$\boldsymbol{A}_s := \boldsymbol{A} - k\boldsymbol{I}_2, \quad \boldsymbol{A} := \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix},$$

$$\boldsymbol{D} := \mathrm{diag}(d_1, d_2, \dots, d_N).$$

$\boldsymbol{I}_N$ denotes N-dimensional unit matrix. The elements of $\boldsymbol{V}$ and $\boldsymbol{W}$ are given by $\{\boldsymbol{V}\}_{jl} = v_{jl}$ and $\{\boldsymbol{W}\}_{jl} = w_{jl}$, respectively.

3. Stability analysis

For large Ω, the stability of the time-varying system (5) is equivalent to that of the following time-invariant system [15]:

$$\dot{\boldsymbol{X}}(t) = (\boldsymbol{I}_N \otimes \boldsymbol{A}_s)\,\boldsymbol{X}(t) + k(\boldsymbol{D}^{-1} \otimes \boldsymbol{I}_2)\left\{\frac{(\boldsymbol{V} \otimes \boldsymbol{I}_2)}{2\delta}\int_{t-\tau_0-\delta}^{t-\tau_0+\delta}\boldsymbol{X}(\theta)\mathrm{d}\theta + (\boldsymbol{W} \otimes \boldsymbol{I}_2)\boldsymbol{X}_{\tau_0}\right\}. \quad (6)$$

The stability of the time-invariant system (6) is governed by the characteristic equation,

$$\begin{aligned} G(s) &= \det\Big[s\boldsymbol{I}_{2N} - (\boldsymbol{I}_N \otimes \boldsymbol{A}_s) - k(\boldsymbol{D}^{-1} \otimes \boldsymbol{I}_2)\{(\boldsymbol{V} \otimes \boldsymbol{I}_2)e^{-s\tau_0}H(s\delta) + (\boldsymbol{W} \otimes \boldsymbol{I}_2)e^{-s\tau_0}\}\Big] \\ &= \det\Big[s\boldsymbol{I}_{2N} - (\boldsymbol{I}_N \otimes \boldsymbol{A}_s) - k\,(\boldsymbol{M} \otimes \boldsymbol{I}_2)\,e^{-s\tau_0}\Big] = 0, \end{aligned} \quad (7)$$

where

$$\boldsymbol{M} := \boldsymbol{D}^{-1}(\boldsymbol{V}H(s\delta) + \boldsymbol{W}), \quad H(x) := \begin{cases} \frac{\sinh x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}.$$

The matrix $\boldsymbol{M}$ can be diagonalized[1] as $\boldsymbol{T}^{-1}\boldsymbol{M}\boldsymbol{T} = \mathrm{diag}(\rho_1(s), \rho_2(s), \dots, \rho_N(s))$ by a matrix $\boldsymbol{T}$, where $\rho_p(s)$ ($p = 1, \dots, N$) are the eigenvalues[2] of $\boldsymbol{M}$. Thus, characteristic equation (7) can be rewritten as

$$\begin{aligned} G(s) &= \det\Big[s\boldsymbol{I}_{2N} - (\boldsymbol{I}_N \otimes \boldsymbol{A}_s) - k\,\{\mathrm{diag}\,(\rho_1(s), \dots, \rho_N(s)) \otimes \boldsymbol{I}_2\}\,e^{-s\tau_0}\Big] \\ &= \prod_{p=1}^{N} g\left(s, \rho_p(s)\right) = 0, \end{aligned}$$

where

$$g\,(s, \rho) := \{s - \omega + k\,(1 - \rho e^{-s\tau_0})\}^2 + \omega^2. \quad (8)$$

The steady state (3) is stable if and only if all the roots of $g\left(s, \rho_p(s)\right) = 0$ lie in the open left-half complex plane for every $\rho_p(s)$ ($p = 1, 2, \dots, N$).

Now, we derive the death region on the connection parameter space. Substituting $s = i\lambda_I$ ($\lambda_I \in \mathbb{R}$) into Eq. (8), its real and imaginary parts are described by

$$\mathrm{Re}\,[g\,(i\lambda_I, \rho(i\lambda_I))] = -\mu + k - k\rho(i\lambda_I)\cos(\lambda_I\tau_0) = 0,$$

$$\mathrm{Im}\,[g\,(i\lambda_I, \rho(i\lambda_I))] = \lambda_I - \omega + k\rho(i\lambda_I)\sin(\lambda_I\tau_0) = 0. \quad (9)$$

From $\sin^2(x) + \cos^2(x) = 1$, we obtain

$$\left(1 - \rho(i\lambda_I)^2\right)k^2 - 2\mu k + \mu^2 + (\omega - \lambda_I)^2 = 0, \quad (10)$$

which does not depend on τ_0. The solution $k(\lambda_I)$ of Eq. (10) is calculated as

$$k\,(\lambda_I) = \frac{\mu \pm \sqrt{D\,(\lambda_I)}}{1 - \rho(i\lambda_I)^2},$$

$$D\,(\lambda_I) := \rho(i\lambda_I)^2\mu^2 + (\rho(i\lambda_I) - 1)\,(\omega - \lambda_I)^2. \quad (11)$$

There exists two solutions $k(\lambda_I)$ for $D\,(\lambda_I) > 0$. Moreover, from Eq. (9), we have

$$\tau_0(\lambda_I, n) = \begin{cases} (\overline{\tau}_0(\lambda_I) + 2n\pi)/\lambda_I & \text{if } \frac{k(\lambda_I)-\mu}{k(\lambda_I)\rho(i\lambda_I)} > 0 \\ (\overline{\tau}_0(\lambda_I) + (2n+1)\pi)/\lambda_I & \text{if } \frac{k(\lambda_I)-\mu}{k(\lambda_I)\rho(i\lambda_I)} < 0 \end{cases}, \quad (12)$$

[1] This is because $\boldsymbol{M}$ and real symmetric matrix $\hat{\boldsymbol{M}} := \boldsymbol{D}^{-1/2}(\boldsymbol{V}H(s\delta) + \boldsymbol{W})\boldsymbol{D}^{-1/2}$ are similar and their eigenvalues $\rho(s)$ are real values.

[2] The elements of $\boldsymbol{M}$ include $H(s\delta)$, then its eigenvalues $\rho(s)$ are functions of s.

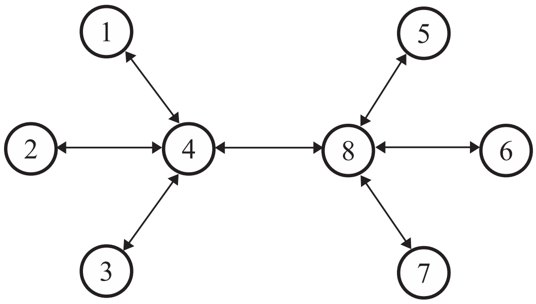
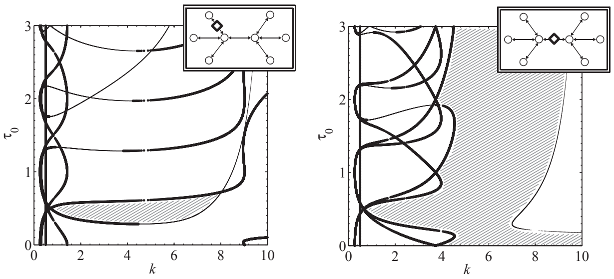


Figure 2: Network oscillators ($N = 8$)



(a) non-hub ∼ hub (b) hub ∼ hub

Figure 3: Stability region with one time-varying delay connection. The symbol ◇ denotes the time-varying delay connection.

where

$$\overline{\tau}_0(\lambda_I) := \tan^{-1}\left(\frac{\omega - \lambda_I}{k(\lambda_I) - \mu}\right), \quad n \in \mathbb{Z}.$$

The marginal stability curve can be derived on $k - \tau_0$ space by using $k(\lambda_I)$ in Eq. (11) and $\tau_0(\lambda_I, n)$ in Eq. (12) for λ_I in the range $D(\lambda_I) > 0$. Furthermore, the sign of the real part of $\mathrm{d}s/\mathrm{d}k$ determines the direction of roots crossing the imaginary axis:

$$\mathrm{Re}\left[\frac{\mathrm{d}s}{\mathrm{d}k}\right]_{\substack{s=i\lambda_I \\ k=k(\lambda_I)}}$$

$$= \mathrm{Re}\left[\frac{1 - \rho(i\lambda_I)e^{-i\lambda_I\tau_0(\lambda_I,n)}}{k(\lambda_I)e^{-i\lambda_I\tau_0(\lambda_I,n)}\left(\left.\frac{\mathrm{d}\rho}{\mathrm{d}s}\right|_{s=i\lambda_I} - \rho(i\lambda_I)\tau_0(\lambda_I, n)\right) - 1}\right]. \quad (13)$$

When the connection parameter set (k, τ_0) crosses the marginal stability curve with increasing k, the positive (negative) value of Eq. (13) means that a root crosses the imaginary axis from left to right (right to left).

4. Numerical example

Let us consider eight oscillators as illustrated in Fig. 2. The 4-th and 8-th oscillators have high degree $d_4 = d_8 = 4$ and the others have low degree $d_1 = d_2 = d_3 = d_5 = d_6 = d_7 = 1$. Let us define the high-degree (low-degree) oscillators as hub (non-hub) oscillators. The parameters of all the oscillators are fixed at $\mu = 0.5$ and $\omega = \pi$, and the delay amplitude is set to $\delta = 1$. In order to investigate the influence of the number of the time-varying delay connections and their configuration on amplitude death, we shall consider the following situations: one (two) time delay(s) is (are) varied and the others are maintained constant.

4.1. One time-varying delay connection

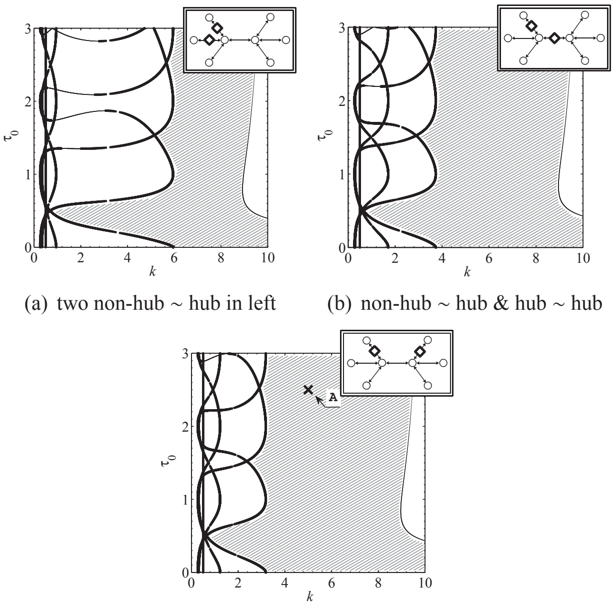
This subsection considers the situation with one time-varying delay connection and with six time-invariant delay connections. The symbol $i \sim j$ denotes that the i-th oscillator is connected to the j-th oscillator by the time-varying delay connection. Let us consider two cases: (i) non-hub ∼ hub connection, (ii) hub ∼ hub connection (i.e. $4 \sim 8$). Without loss of generality, we employ $1 \sim 4$ for case (i).

The stability regions for cases (i) and (ii) are shown in Figs. 3(a) and 3(b), respectively. The marginal stability curves denote the solution of $g(i\lambda_I, \rho(i\lambda_I)) = 0$. When the connection parameter set (k, τ_0) crosses the bold (thin) curve with increasing k, a root of $g(s, \rho(s)) = 0$ crosses the imaginary axis from right to left (left to right). The shaded area is the stability region where all the roots of $g(s, \rho(s)) = 0$ lie in the open left-half complex plane.

As shown in Fig. 3(a), there is the small stability region. This fact implies that we cannot induce amplitude death for long delay time τ_0. On the contrary, in Fig. 3(b), there is the large stability region and exist no curves in the range $k \in (4.545, 7.011)$ on the $k-\tau_0$ space. We see no upper limit of τ_0; thus, amplitude death can be induced for arbitrarily long nominal delay time τ_0 within the range. From these results, for one time-varying delay connection, we should vary the time delay in the connection between the two hub oscillators to obtain a large stability region.

4.2. Two time-varying delay connections

Now we consider the situation with two time-varying delay connections and with five time-invariant delay connections. Let us focus on three cases: (i) two non-hub ∼ hub connections in the left or right side (Fig. 4(a)), (ii) non-hub ∼ hub connection and hub ∼ hub connection (Fig. 4(b)), (iii) non-hub ∼ hub connection in both sides (Fig. 4(c)). Without loss of generality, we employ $1 \sim 4$ and $2 \sim 4$ for case (i), $1 \sim 4$ and $4 \sim 8$ for case (ii), and $1 \sim 4$ and $5 \sim 8$ for case (iii). Figures 4(a), 4(b), and 4(c) show the stability regions for cases (i), (ii), (iii), respectively. All the stability regions in Fig. 4 have the ranges k where amplitude death is induced for arbitrarily long nominal delay time τ_0. It can be seen that the region in Fig. 4(a) is smaller than that in Fig. 4(b). From this result, one may conclude that the time-varying delay connection between the two hub oscillators plays an important role in expanding the region as is the case with Fig. 3. This conclusion is not always true. As shown in Fig. 4(c), we have the region nearly as large as



(a) two non-hub ~ hub in left   (b) non-hub ~ hub & hub ~ hub

(c) non-hub ~ hub in both sides

Figure 4: Stability region with two time-varying delay connections. The symbol ◇ denotes the time-varying delay connection.

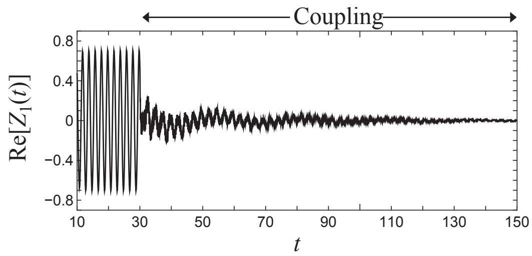


Figure 5: Time-series data of the 1st oscillator at point A in Fig. 4(c)

that in Fig. 4(b) even though the delay time in connection between the two hub oscillators is maintained constant.

Figure 5 shows the time-series data of the first oscillator at point A $(k, \tau_0) = (5, 2.5)$ in Fig. 4(c). The frequency variation Ω is set to a large value $\Omega = 10\pi$. Eight oscillators are coupled at $t = 30$. After coupling, the state variable $\mathrm{Re}[Z_1(t)]$ converges on the fixed point.

5. Conclusion

This study analyzed the stability of amplitude death induced by the partial time-varying delay connection. This connection consists of time-invariant delay connections and time-varying delay connections. Our results suggest that the partial time-varying delay connection is useful for death induction. The analytical results were confirmed by numerical simulations.

Acknowledgments

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