2007 International Symposium on Nonlinear Theory and its Applications
NOLTA'07, Vancouver, Canada, September 16-19, 2007

Nonresonant entrainment between detuned oscillators induced by common external noise

Kazuyuki Yoshimura†, Peter Davis†, and Atsushi Uchida‡

†NTT Communication Science Laboratories, NTT Corporation
2-4, Hikaridai, Seika-cho, Soraku-gun, Kyoto 619-0237, Japan
‡Department of Electronics and Computer Systems, Takushoku University
815-1 Tatemachi, Hachioji, Tokyo 193-0985, Japan
Email: kazuyuki@cslab.kecl.ntt.co.jp, davis@cslab.kecl.ntt.co.jp, a-uchida@es.takushoku-u.ac.jp

Abstract—We theoretically and numerically studied entrainment of two uncoupled nonidentical limit cycle oscillators subjected to a common external white Gaussian noise. We have found that a novel type of entrainment occurs in a general class of oscillators. In this entrainment, the mean frequency difference of the two oscillators remains the same as their natural frequency difference but their phases come to be locked with each other for almost all the time as the noise intensity increases. This absence of frequency locking is a fundamental property of noise induced entrainment.

1. Introduction

Entrainment is a key mechanism for the emergence of order and coherence in a variety of physical systems consisting of oscillatory elements. It is important to explore the possible types of entrainments and clarify their fundamental properties. It is well known that a common external periodic input may lead to entrainment between two independent and slightly detuned oscillators when their natural frequencies are in resonance with the input frequency (e.g., see [1]). A fundamental property of this resonant entrainment is that both frequencies and phases of the two oscillators lock with each other.

Recent physical and numerical experiments have shown that not only a periodic but also a noise-like signal can give rise to entrainment between two independent oscillators [2, 3, 4, 5]. Experimental evidence for this phenomenon has been found for several systems as diverse as neuronal networks [2], ecological systems [3], and lasers [4]. The entrainment by a noise-like signal is a nonresonant one in the sense that there is no resonance relation between the oscillator and the noise. Therefore, we call it *nonresonant entrainment*. It is essential for fully understanding the emergence of order in the real world to clarify fundamental properties of this nonresonant entrainment.

The nonresonant entrainment between two independent oscillators with an *identical natural frequency* has been studied [5]. It has been shown that a stable phase locking is achieved and the two oscillators have the same mean frequency when a weak Gaussian noise is applied. In this sense, the behavior is similar to that for periodic input, and seems to support the view there is no fundamental difference between resonant and nonresonant entrainment.

In real systems, two oscillators are in general slightly detuned. Thus, it is crucial to consider the case of detuned oscillators. We considered two independent and slightly detuned limit cycle oscillators subjected to a common white Gaussian noise and have clarified fundamental properties of nonresonant entrainment, which are quite different from those of resonant entrainment. Our theoretical results apply to a general class of limit cycle oscillators.

2. Theory

Let $\boldsymbol{X}_i \in \mathbf{R}^N$ be a state variable vector and consider the equation

$$\dot{\boldsymbol{X}}_i = \boldsymbol{F}(\boldsymbol{X}_i) + \delta\boldsymbol{F}_i(\boldsymbol{X}_i) + \boldsymbol{G}(\boldsymbol{X}_i)\eta(t), \quad i = 1, 2, \qquad (1)$$

where $\boldsymbol{F}$ is an unperturbed vector field, $\delta\boldsymbol{F}_1$ and $\delta\boldsymbol{F}_2$ are small deviations from it, $\boldsymbol{G} \in \mathbf{R}^N$ is a vector function, and $\eta(t)$ is the white Gaussian noise such that $\langle\eta(t)\rangle = 0$ and $\langle\eta(t)\eta(s)\rangle = 2D\,\delta(t-s)$, where $\langle\cdots\rangle$ denotes averaging over the realizations of η and δ is Dirac's delta function. We call the constant $D > 0$ the noise intensity. The noise-free unperturbed system $\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X})$ is assumed to have a limit cycle with a frequency ω. We employ the Stratonovich interpretation for the stochastic differential equation (1). This interpretation allows us to apply the phase reduction method to Eq. (1), which assumes the conventional variable transformations in differential equations.

If we regard the common noise as a weak perturbation to the deterministic oscillators and apply the phase reduction method to Eq. (1), we obtain the equation for the phase variable as follows:

$$\dot{\phi}_i = \omega + \delta\omega_i(\phi_i) + Z(\phi_i)\eta(t), \quad i = 1, 2, \qquad (2)$$

where ω is the frequency of the unperturbed oscillator, $\delta\omega_i$ is the frequency variation due to $\delta\boldsymbol{F}_i$, Z is defined by $Z(\phi) = \boldsymbol{G}(\boldsymbol{X}_0(\phi)) \cdot \left(\mathrm{grad}_x\phi|_{x=x_0(\phi)}\right)$, where ϕ is the phase variable defined by the unperturbed system $\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X})$ and $\boldsymbol{X}_0(\phi)$ is its limit cycle solution. By definition, $Z(\phi)$ is a periodic function, i.e., $Z(\phi) = Z(\phi + 2\pi)$. We assume that Z is three times continuously differentiable and not a constant. It is also assumed that $0 < D/\omega \ll 1$ to ensure the validity of the phase reduction.

In order to derive the average equation for ϕ_i, we translate Eq. (2) into the equivalent Ito stochastic differential equation:

$$\dot{\phi}_i = \omega + \delta\omega_i(\phi_i) + DZ(\phi_i)Z'(\phi_i) + Z(\phi_i)\eta(t), \qquad (3)$$

where the dash denotes differentiation with respect to ϕ_i. In the Ito equation, unlike in Stratonovich formulation, the correlation between ϕ_i and η vanishes. If we subtract Eq. (3) for ϕ_2 from that for ϕ_1 and take the ensemble average, then we have the average equation

$$\begin{aligned}\frac{d}{dt}\langle\phi_1 - \phi_2\rangle &= \langle\delta\omega_1(\phi_1)\rangle - \langle\delta\omega_2(\phi_2)\rangle \\ &+ D\{\langle Z(\phi_1)Z'(\phi_1)\rangle - \langle Z(\phi_2)Z'(\phi_2)\rangle\}, \quad (4)\end{aligned}$$

where we used the fact $\langle Z(\phi_i)\eta(t)\rangle = \langle Z(\phi_i)\rangle\langle\eta(t)\rangle = 0$. Each ensemble average on the right hand side can be evaluated by using the steady probability distribution $P_i(\phi_i)$ for ϕ_i, which can be obtained from the Fokker-Planck equation for Eq. (3): i.e., $\langle A(\phi_i)\rangle = \int_0^{2\pi} A(\phi)P_i(\phi)d\phi$, where A represents a function of ϕ_i. The distribution P_i can be obtained as $P_i(\phi_i) = 1/2\pi + O(\sigma_i, D/\omega)$, where $\sigma_i = \max_{0\le\phi<2\pi}|\delta\omega_i(\phi)/\omega|$. Since $\delta\boldsymbol{F}_i$ is small, σ_i is a small parameter. Therefore, P_i can be approximated by $P_i \simeq 1/2\pi$ for small D/ω and we have $\langle\delta\omega_i(\phi_i)\rangle \simeq \int_0^{2\pi}\delta\omega_i(\phi)/2\pi\, d\phi \equiv \overline{\delta\omega_i}$ and $\langle Z(\phi_i)Z'(\phi_i)\rangle \simeq \int_0^{2\pi} Z(\phi)Z'(\phi)/2\pi\, d\phi = 0$. If we substitute these results into Eq. (4), we have

$$\frac{d}{dt}\langle\phi_1 - \phi_2\rangle = \overline{\delta\omega_1} - \overline{\delta\omega_2}. \qquad (5)$$

Since in general $\overline{\delta\omega_1} - \overline{\delta\omega_2} \ne 0$, this equation indicates that the average phase difference increases or decreases in proportion to the time t. The two oscillators have different mean frequencies even when a common white Gaussian noise is applied, i.e., $d\langle\phi_1\rangle/dt \ne d\langle\phi_2\rangle/dt$. Equation (5) shows that the mean frequency difference is independent of the noise intensity and it equals the natural frequency difference. Intuitively, this result is natural because the white noise has a uniform power spectrum and does not have a characteristic frequency, which could entrain the oscillator frequencies.

Let θ and ψ be defined by $\theta = \phi_1 - \phi_2$ and $\psi = \phi_1 + \phi_2 - 2\omega t$. The variable θ measures the phase difference between the two oscillators. For small D and $\overline{\delta\omega_i}$, it is expected that ϕ_i still has a mean frequency close to ω. Therefore, θ and ψ can be regarded as slow variables. If we change the independent variables form (t, ϕ_1, ϕ_2) to (t, θ, ψ) and perform the time-averaging with respect to t, we can obtain the Fokker-Planck equation corresponding to Eq. (2) as follows:

$$\begin{aligned}\frac{\partial Q}{\partial t} &= -(\overline{\delta\omega_1} + \overline{\delta\omega_2})\frac{\partial Q}{\partial\psi} - (\overline{\delta\omega_1} - \overline{\delta\omega_2})\frac{\partial Q}{\partial\theta} \\ &+ D\frac{\partial^2}{\partial\psi^2}[\,g(\theta)Q\,] + D\frac{\partial^2}{\partial\theta^2}[\,h(\theta)Q\,] \qquad (6)\end{aligned}$$

where $Q(t, \theta, \psi)$ is the joint probability distribution. The functions g and h are given by $g(\theta) = 2\{\Gamma(0) + \Gamma(\theta)\}$ and $h(\theta) = 2\{\Gamma(0) - \Gamma(\theta)\}$, where Γ is defined by $\Gamma(\theta) = (2\pi)^{-1}\int_0^{2\pi} Z(\phi)Z(\phi + \theta)d\phi$. Hereafter we assume the case of $\overline{\delta\omega_1} > \overline{\delta\omega_2}$ without loss of generality.

It is in general possible that Z has a period smaller than 2π. Since Z is not a constant function, we suppose that $Z(\phi) = Z(\phi + 2\pi/n)$, where n is a positive integer. It can be shown that $h(\theta) \ge 0$ for any $\theta \in [0, 2\pi)$. The zero points s_m of h are given by $s_m = 2\pi m/n,\ m = 0, 1, \ldots, n-1$, where $s_0 = 0$. Equation (6) has the steady solution $Q_s(\theta)$ such that it is a continuous function of θ only and satisfies the two conditions (i) $Q_s(\theta) = Q_s(\theta + 2\pi)$ and (ii) $\int_0^{2\pi} Q_s(\theta)d\theta = 1$. In each interval (s_m, s_{m+1}), the solution Q_s can be obtained as follows:

$$Q_s(\theta) = \frac{\varepsilon}{2\pi h(\theta)}\int_\theta^{s_{m+1}} \exp\left[-\varepsilon\int_\theta^x \frac{1}{h(y)}dy\right]dx, \qquad (7)$$

where $\varepsilon = (\overline{\delta\omega_1} - \overline{\delta\omega_2})/D > 0$. The right hand side of Eq. (7) has singularities at the zero points of h. The value of Q_s for each s_m is given by $Q_s(s_m) = \lim_{\theta\to s_m} Q_s(\theta)$. Assume that $\theta \in (s_m, s_{m+1})$, i.e., θ is an arbitrary regular point. It can be shown that $\lim_{\varepsilon\to 0} Q_s(\theta) = 0$ holds due to the factor ε in the numerator. This implies that the probability has to concentrates at the singular points $s_m,\ m = 0, 1, \ldots, n-1$ because Q_s satisfies the condition (ii). Thus, Q_s in the limit $\varepsilon \to 0$ is given by

$$Q_s(\theta) = \frac{1}{n}\sum_{m=0}^{n-1}\delta(\theta - s_m), \qquad (8)$$

where δ is Dirac's delta function. For small positive ε, the distribution Q_s has narrow and sharp peaks at $\theta = s_m,\ m = 0, 1, \ldots, n-1$ while Q_s is close to zero in the regions other than the neighborhoods of these singular points. The peaks of Q_s become narrower and higher as ε approaches zero. Equation (8) indicates that multiple peaks exist if Z has a period smaller than 2π, i.e., $n > 1$.

The above profile of Q_s clearly shows that the phase locking states, where $\theta \simeq s_m$ (mod 2π), are achieved for a large fraction of time during the time evolution when the noise intensity D is relatively large with respect to the natural frequency difference $\overline{\delta\omega_1} - \overline{\delta\omega_2}$: i.e., the nonresonant entrainment occurs. Let δ be a small positive constant and U_δ be the δ-neighborhood defined by $U_\delta = \cup_{m=0}^{n-1}(s_m - \delta, s_m + \delta)$, where mod 2π is taken for $s_0 - \delta$. We identify the phase locking state by the condition $\theta \in U_\delta$. As shown by Eq. (5), the present entrainment is not characterized by coincidence of the mean frequencies of the two oscillators. Therefore, as a measure for the entrainment, we introduce the phase locking time ratio μ defined by

$$\mu = \lim_{T\to\infty}\frac{T_L}{T}, \qquad (9)$$

where T_L represents the total time length for which $\theta \in U_\delta$ happens during the period T. This ratio can also be expressed in terms of Q_s by $\mu = \int_{U_\delta} Q_s(\theta)d\theta$, where the integral is taken over the set U_δ. Equation (8) shows that $\mu \to 1$ in the limit $\varepsilon = (\overline{\delta\omega}_1 - \overline{\delta\omega}_2)/D \to 0$.

A phase locking state cannot continue for the infinite time but phase slips have to happen during the periods such that $\theta \notin U_\delta$ because the two mean frequencies $d\langle\phi_1\rangle/dt$ and $d\langle\phi_2\rangle/dt$ are different. Equation (5) indicates that the mean frequency difference is given by the constant $\overline{\delta\omega}_1 - \overline{\delta\omega}_2$. This implies that the average number of phase slips, which happen in a unit time interval, does not become small but remains constant even for relatively large D compared with $\overline{\delta\omega}_1 - \overline{\delta\omega}_2$. In other words, the average interslip interval remains constant. On the other hand, the probability for $\theta \notin U_\delta$ decreases and converges to zero as D increases: i.e., the phases come to be locked for almost all the time. These two facts imply that a single phase slip completes more rapidly: i.e., the time needed for one phase slip decreases and converges to zero as D increases. We emphasize that the above mentioned behavior is a remarkable feature of the nonresonant entrainment. This behavior is very different from that of resonant entrainment by a periodic signal, where the average interslip interval diverges and the mean frequencies become identical as the signal intensity approaches the critical value for entrainment.

3. Numerical examples

In order to demonstrate the above analytical results, we show numerical results for an example described by the Stratonovich stochastic differential equations

$$\dot{\phi}_j = \omega_j + \sin(\phi_j)\,\eta(t), \quad j = 1, 2, \tag{10}$$

where ω_j, $j = 1, 2$ are slightly different constants.

Figure 1(a) shows the mean frequency difference $\Delta\omega = d\langle\phi_1\rangle/dt - d\langle\phi_2\rangle/dt$ plotted as a function of D, where ω_1 is fixed to unity and five different values of ω_2 are employed. The mean frequency difference $\Delta\omega$ is not zero except for the case $\omega_1 = \omega_2 = 1$. It is clearly shown that $\Delta\omega$ is constant and independent of D. This coincides with the analytical result of Eq. (5) . The steady distribution $P_j(\phi_j)$ is approximately given by $P_j(\phi_j) \simeq (1/2\pi)[\,1 + (D/2\omega_j)\sin(2\phi_j)\,]$ for this example. This shows that the assumption $P_j(\phi_j) \simeq 1/2\pi$ is reasonable for small D used in the numerical calculations. Thus, the result of Eq. (5) holds.

The time evolution of the phase difference $\theta = \phi_1 - \phi_2$ is shown for three different values of D in Fig. 1(b), where $\omega_1 = 1$ and $\omega_2 = 0.98$. These results clearly show that the phases are locked near $\theta \simeq 2\pi n$, $n \in \mathbf{Z}$ and the phase slips occur intermittently. It should be noted that the time needed for a single phase slip becomes smaller as D increases. This observation is in agreement with the analytical result.

The probability distribution $Q_s(\theta)$ is shown in Fig. 1(c) for three different values of D, where $\omega_1 = 1$ and $\omega_2 =$

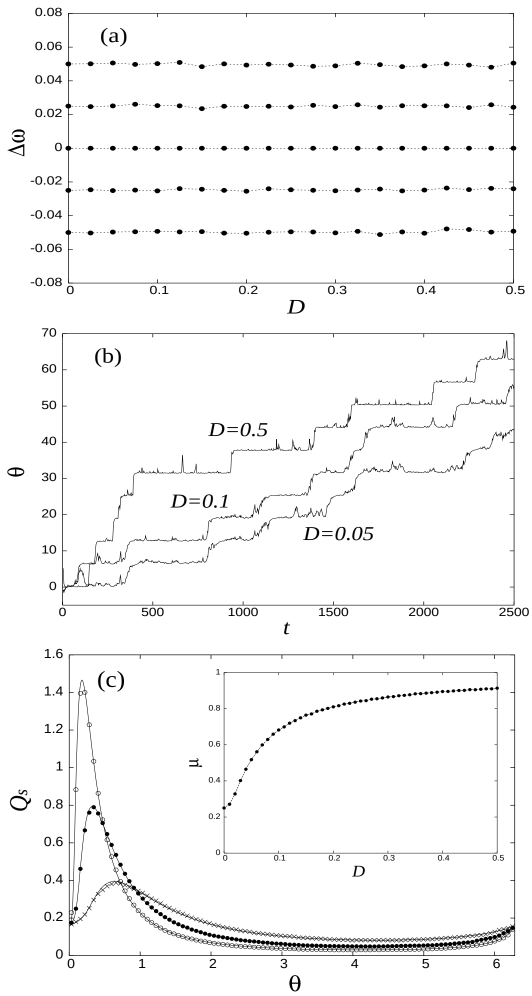

Figure 1: Entrainment in phase models with $Z = \sin(\phi)$: (a) mean frequency difference $\Delta\omega$ vs. D, (b) time evolution of phase difference $\theta = \phi_1 - \phi_2$, and (c) probability distribution $Q_s(\theta)$ for $D = 0.02\,(\times)$, $0.05\,(\bullet)$, and $0.1\,(\circ)$, where analytical results are shown by solid line. The inset shows μ plotted against D. In (b) and (c), $\omega_1 = 1$ and $\omega_2 = 0.98$.

0.98. The analytical results of Eq. (7) are also shown for the corresponding values of $\varepsilon = (\omega_1 - \omega_2)/D$. The numerical results are in good agreement with the analytical one. It is clearly seen that Q_s has a sharp peak near $\theta = 0$ for large D or small ε. The peak in Q_s becomes narrower and its position becomes closer to $\theta = 0$ as D increases. It is also seen that the peak is not centered at $\theta = 0$ but shifted to the positive direction: i.e., the phase ϕ_1 of the larger natural frequency oscillator is kept advanced with respect to ϕ_2 even in the phase locking state. The inset of Fig. 1(c) shows that the phase locking time ratio μ monotonically increases and approaches unity with increasing D. Figure 1(c) clearly demonstrates that the phases are locked for a larger fraction of the time as D increases.

In order to validate the theory based on the phase reduction method, we carried out numerical experiments for the

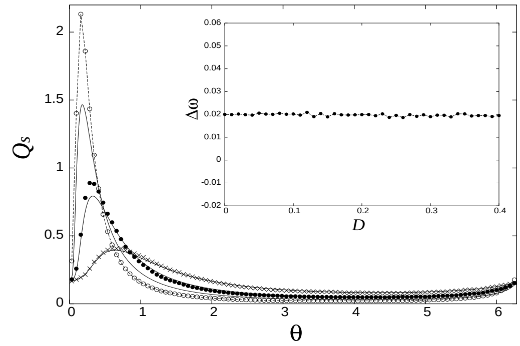

Figure 2: Entrainment in SL oscillators. Probability distribution $Q_s(\theta)$ is shown for $D = 0.02\,(\times)$, $0.05\,(\bullet)$, and $0.1\,(\circ)$. Analytical results are also shown by solid line. The inset shows $\Delta\omega$ vs. D. Parameters are $\delta\omega_1 = 0$ and $\delta\omega_2 = -0.02$.

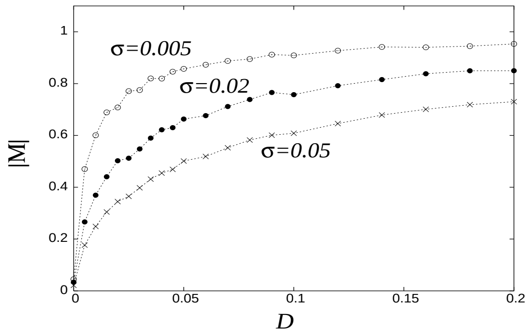

Figure 3: Order parameter $|M|$ vs. D for $\omega_0 = 1$ and $\sigma = 0.005\,(\circ)$, $0.02\,(\bullet)$, and $0.05\,(\times)$.

Stuart-Landau (SL) oscillator

$$\dot{\psi}_j = (1 + ic_j)\psi_j - |\psi_j|^2\psi_j - \eta(t), \quad j = 1, 2, \tag{11}$$

where $\psi_j \in \mathbf{C}$ and $c_j = 1 + \delta\omega_j$ is a real constant. This is reduced to the phase model $\dot{\phi}_j = 1 + \delta\omega_j + \sin(\phi_j)\eta(t)$, where ϕ_j is the appropriately defined phase variable.

In Fig. 2, the numerically obtained distribution $Q_s(\theta)$ is shown for three different values of D, where $\delta\omega_1 = 0$ and $\delta\omega_2 = -0.02$. The analytical results obtained from the corresponding phase model are also shown for the corresponding values of $\varepsilon = (\delta\omega_1 - \delta\omega_2)/D$. A sharp peak of Q_s appears near $\theta = 0$. It becomes narrower and approaches $\theta = 0$ as D increases. Agreement between the numerical and analytical results is excellent, especially in small D region, where the phase reduction method gives a good approximation. The inset shows the mean frequency difference $\Delta\omega = d\langle\phi_1\rangle/dt - d\langle\phi_2\rangle/dt$ plotted as a function of D for the same $\delta\omega_1$ and $\delta\omega_2$. It is clearly shown that $\Delta\omega$ does not depend on D and its constant value is given by $\Delta\omega = \delta\omega_1 - \delta\omega_2$. This behavior also agrees with the theory. The agreements in the behaviors of Q_s and $\Delta\omega$ validate the theory based on the phase model.

4. Emergence of macroscopic rhythm

Nonresonant entrainment can induce macroscopic rhythm in an ensemble of detuned oscillators. To demonstrate this, consider an ensemble of N oscillators given by Eq. (10), in which the natural frequencies ω_j distribute according to the Gaussian distribution $P(\omega) = \exp[-(\omega - \omega_0)^2/2\sigma^2]/\sqrt{2\pi}\sigma$. We define the order parameter M by $M = (1/N)\sum_{j=1}^{N}\exp[i\phi_j]$ to measure the intensity of collective motion: if the phases evolve collectively, $|M|$ approaches unity. Figure 3 shows $|M|$ plotted against D, where $\omega_0 = 1$ and $\sigma = 0.005,\ 0.02,$ and 0.05. It shows that $|M|$ increases and becomes close to unity as D increases, indicating the emergence of collective motion. It was observed that M oscillates almost periodically for values of D used in Fig. 3. Thus, a common noise can generate a macroscopic rhythm via nonresonant entrainment in an ensemble of detuned oscillators.

5. Conclusions

In conclusion, we have revealed the nature of nonresonant entrainment, considering two detuned limit cycle oscillators subjected to a common external white Gaussian noise. We theoretically elucidated this phenomenon by using a phase model and presented numerical evidence for a particular phase model and the SL oscillator. We found that the mean frequency difference of the two oscillators remains identical with their natural frequency difference but their phases come to be locked with each other for almost all the time as the noise intensity increases. This is a fundamental property of nonresonant entrainment.

References

[1] Y. Kuramoto, *Chemical Oscillation, Waves, and Turbulence* (Springer-Verlag, Tokyo, 1984).

[2] Z. F. Mainen and T. J. Sejnowski, "Reliability of spike timing in neocortical neurons," *Science*, vol.268, pp.1503–1506, 1995.

[3] B. T. Grenfell *et al.*, "Noise and determinism in synchronized sheep dynamics," *Nature*, vol.394, pp.674–677, 1998.

[4] T. Yamamoto, I. Oowada, H. Yip, A. Uchida, S. Yoshimori, K. Yoshimura, J. Muramatsu, Shin-itiro Goto, and P. Davis, "Common-chaotic-signal induced synchronization in semiconductor lasers," *Optics Express*, vol.15, pp.3974–3980, 2007.

[5] J. Teramae and D. Tanaka, *Phys. Rev. Lett.*, vol.93, p.204103, 2004; H. Nakao, K. Arai, and Y. Kawamura, *Phys. Rev. Lett.*, vol.98, p.184101, 2007.