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The analysis of phase coherence of an ensemble of uncoupled limit-cycle oscillators via averaging of a jump-diffusion Kolmogorov equation

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Abstract—Identical uncoupled limit-cycle oscillators receiving common noise show various patterns of coherence, including complete desynchronization, phase clustering, and phase synchronization. We extend the averaging method employed earlier to analyze the global coherence properties of limit-cycle oscillators receiving weak Gaussian white-noise [1] to the case where the common noise is compound random Poisson impulses.

1. INTRODUCTION

The coherence of members of a population receiving a common input models many real-world systems such as the synchronization of various population and fruiting patterns of plants and animals [2], the reproducibility of laser intensity when driven by a common drive signal [3, 4], and the reproducibility of spike trains from identical neurons receiving identical fluctuating synaptic current [5]. Collective coherence in nature often manifest themselves to casual observers as rhythms and patterns that form despite the seemingly random external forces that act upon the system. A method to control such collective properties through random external forcing would therefore have important applications in the control of complex dynamical systems.

In the realm of theoretical investigation, the phenomenon of phase coherence due to common noise applied to an ensemble of oscillating elements has been the subject of research for some time. Originally, the phenomenon was discovered in chaotic oscillators, but a quantitative theory has not been found to explain the phenomenon [6, 7, 8]. A theory showing synchronization has been developed for uncoupled limit-cycle oscillators receiving additive weak Gaussian white-noise [9]. Removing the restriction of additivity allows the formation of clusters [1], and the removing the restriction of infinitesimal signal strengths adds de-synchronization to the repertoire of ensemble behavior [10, 11]. Non-identicality of the oscillators has also been addressed [12], and gives rise to imperfect entrainment characterized by intermittent failure of a phase locked state. In the previous works on the synchronization phenomena of limit-cycle oscillators, the stability of the synchronized state was analyzed by considering the local Lyapunov exponents of the separation between two oscillators. The only piece of information necessary to assess the stability was the shape of the phase response curve (PRC) [13] of the oscillators. The utility of this is obvious, because for many systems of interest, it would not be possible to deduce the dynamical equations of the system. The PRC is a relatively easy function to obtain experimentally without having to know the specific details of the system. In this paper, we analyze coherence properties of the system by obtaining a global distribution function of the phase difference between two oscillators utilizing the PRC as the only piece of information about the dynamical properties of each oscillator.

2. THEORY

2.1. Model

We consider two phase oscillators receiveing a common sequence of random Poisson impulses. The dynamical equation of the phase αth oscillator ($\alpha \in [1, 2]$) is

$$\dot{\theta}_\alpha(t) = \omega_\alpha t + \sqrt{\epsilon}\chi_\alpha(t) + G(\theta_\alpha(t), c_n) \sum_{n=0}^{N} g(t - t_n), \quad (1)$$

$\theta(t) \in [0, 2\pi)$ is the phase of the oscillator at time t, t_n the arrival time of the nth impulse, $g(t - t_n)$ is the common white impulsive noise with a jump of magnitude c_n, and the $\chi_{(\alpha)}(t)$ is assumed to be independent, identically distributed zero-mean Gaussian white noise of unit intensity and with correlation given by $\langle \chi_{i,\alpha}(t)\chi_{j,\beta}(s)\rangle = \delta_{\alpha,\beta}\delta_{i,j}\delta(t - s)$. The map $G(\phi, c)$ is the PRC. Given the impulse strength and the phase at which an impulse arrives, it describes the phase-dependent magnitude of the jump experienced by the oscillator.

2.2. The forward-Kolmogorov equation

We rewrite the above equation as a stochastic jump-diffusion equation [14, 15, 16],

$$d\theta_\alpha(t) = \omega_\alpha dt + \sqrt{\epsilon}dW_\alpha(t) + \int_C G(\theta_\alpha(t), c)M(dt, dc), \quad (2)$$

where $dW(t)$ is a white-noise Gaussian increment and $M(dt, dc)$ the Poisson random measure of a marked Poisson point process. The expectations of the stochastic processes are $E[dW(t)] = 0$ and $E[M(dt, dc)] = \lambda p(c)dcdt$, where λ is the rate of the Poisson process and $p(c)$ is the distribution of the random marks. The previous equation is a Langevin equation driven by an additional marked point process, and is generally known as a stochastic differential equation (SDE).

There are two such equations for the above system. There is no coupling, but the two share an identical input from the Poisson impulses and an independent input from the Gaussian noise. In order to investigate the coherence properties of the pair, we analyze the distribution of their phase difference by utilizing the equivalent forward-Kolmogorov equation of the SDE:

$$\frac{\partial P(\phi_1,\phi_2,t)}{\partial t} = \frac{\epsilon}{2}\left(\frac{\partial^2 P}{\partial \phi_1^2}+\frac{\partial^2 P}{\partial \phi_2^2}\right)-\left(\omega_1\frac{\partial P}{\partial \phi_1}+\omega_2\frac{\partial P}{\partial \phi_2}\right)$$
$$-\lambda P(\phi_1,\phi_2,t)+\lambda\int_C P\Big[\phi_1-\eta(\phi_1,c),\phi_2-\eta(\phi_2,c)\Big]\times$$
$$\left|1-\frac{\partial \eta_i}{\partial \phi_j}\right| p(c)dc \qquad (3)$$

where $\frac{\partial \eta_i}{\partial \phi_j} = \frac{\partial \eta}{\partial \phi_j}(\phi_j, c)$, $i, j \in [1,2]$.

The preceeding equation is written as a function of the destination of a jump due to the jump process, $\phi_i(t) = \theta_i(t) + G(\theta_i(t), c)$, and $\eta(\phi_i(t), c) = \phi_i(t) - \theta_i(t)$ is the jump magnitude written as a function of the destination coordinate, $\phi_i(t)$. We assume that $\theta_i(t) + G(\theta_i(t), c)$ is a monotonically increasing function of $\theta_i(t)$. As such, our current theory is applicable only to the situation where common noise induces coherent behavior (synchronization and clustering), because it is known from previous works that if $\theta_i(t) + G(\theta_i(t))$ is monotonically increasing, the locked phases are stable.

2.3. Averaging

The phase distribution function of an unperturbed oscillator is uniform, as the phase is defined to be a constantly increasing variable under such conditions. Random Poisson impulses will, however, give rise to non-uniformity in the distribution [11]. In the limit that $\lambda \to 0$, the single-oscillator phase distribution can be approximated as uniform. Under such assumptions, we may change variables as follows:

$$\psi = \frac{\phi_1+\phi_2}{2}, \qquad \xi = \phi_1 - \phi_2, \qquad (4)$$

and assume that the joint probability density may be decoupled as $P(\phi_1, \phi_2, t) \approx S(\psi)U(\xi)$ with $S(\psi) \approx \frac{1}{2\pi}$. Such a decoupling separates utilizes the fact that one variable is a fast variable (ψ) and one variable is a slow variable (ξ), and that the dynamics of the slow variable can be adequately explained by averaging quantities that depend on the fast variable over a period of the fast variable [17, 1].

The approximate forward-Kolmogorov equation may then be written as a PDE for ξ:

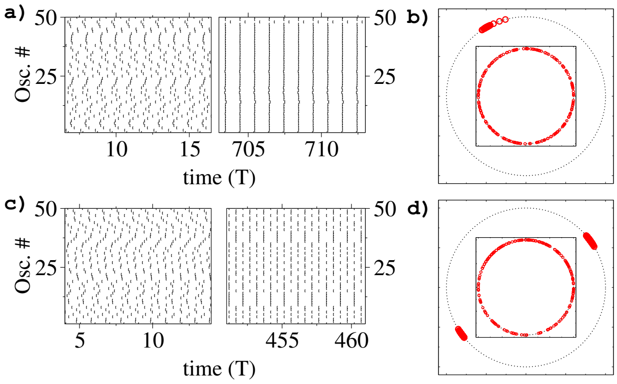

Figure 1: Simulation results of Eq 6. a), c) show the raster plots (tick mark indicates time oscillator passed through phase 0.) of 50 uncoupled phase oscillators receiving common Poisson noise. The plots show initial and final conditions on the left and right, respectively. a) shows the emergence of synchronization due to the common noise for $m = 1$, and c) shows the emergence of two clusters due to the common noise for $m = 2$. b), d) show phase plots of the systems in a) and c), respectively, of a superset of oscillators (200 in both cases). Phase 0 is on the right most point of the circle. The inset shows the intial conditions, while the outer graph shows the emergent coherence due to noise. Common simulation parameters were $\epsilon = 0.002$ and $c = 0.06$.

$$\frac{\partial U}{\partial t}(\xi,t) = \epsilon\frac{\partial^2 U}{\partial \xi^2}(\xi,t) - \lambda U(\xi,t) - (\omega_1-\omega_2)\frac{\partial U}{\partial \xi}(\xi,t)$$
$$+\lambda\int_{\psi_0}^{\psi_1}\int_C U(\xi-\eta(\phi_1,c)+\eta(\phi_2,c),t)\left|1-\frac{\partial \eta_i}{\partial \phi_j}\right|\times$$
$$p(c)dcd\psi,$$
$$\text{where}\begin{cases}\xi \ge 0 & \psi_0 = \frac{\xi}{2}, \psi_1 = 2\pi - \frac{\xi}{2}\\ \xi < 0 & \psi_0 = -\frac{\xi}{2}, \psi_1 = 2\pi + \frac{\xi}{2}\end{cases} \qquad (5)$$

The ξ-dependent limits can be seen when one considers that $\phi_i \in [0, 2\pi)$, $\psi \in [0, 2\pi)$ and $\xi \in [-2\pi, 2\pi)$. The preceding equation is easy to interpret: The time rate of change in the density at the range $[\xi, \xi + d\xi)$ changes due to 1) the first term, which describes the diffusion due to the independent noises added to each oscillator, 2) the second term, which describes the average flux of pairs in the ensemble out of the range $[\xi, \xi + d\xi)$, and 3), the third term, which describes the flux of pairs in the ensemble into the range $[\xi, \xi + d\xi)$. We note that the current approximation is valid only for weak independent noise. We are working on a sufficient theory for all independent noise strengths.

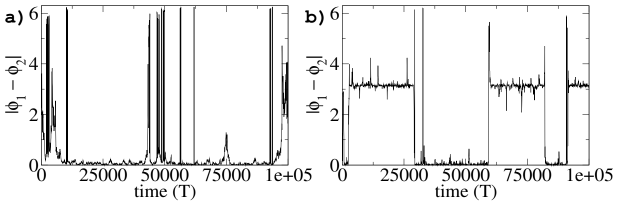

Figure 2: a) Intermittency and b) switching induced by indepedent noise for a pair of oscillators. The time is in units of oscillator period. Since the switching is caused mainly by phase diffusion, the switch is not as sharp and immediate as it may appear here. Intermittency is the occasional failure of synchronization, while switching is intermittency that causes jumps between cluster states.

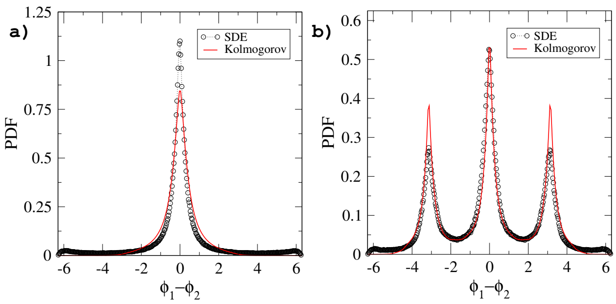

Figure 3: Comparison of $\xi = \phi_1 - \phi_2$ found via direct SDE simulation and iterative calculation of Eq. 5. The width of the peaks of distributions occur due to the imperfect coherence of the coherent states due to the effects of independent noise. Location of peaks agree with previous work that analyzed the stability of the coherent states, but the current work takes into account the independent noise, which in the real world may correspond to thermal noise or non-identicality of the oscillators.

3. Simulation and numerical calculation

We measured the phase distribution function of a pair of identical oscillators described by

$$\dot{\phi_\alpha} = \omega + \sqrt{\epsilon}\eta_\alpha(t) + c\sum_{n=0}^{N} \sin\left(m\phi_\alpha(t)\right) g(t-t_n) \qquad (6)$$

It is known that if the PRC possesses a symmetry within $\phi \in [0, 2\pi)$, which in the above case happens when $m > 1$, m equivalent clustered states arise, with each state possessing the same stability as the synchronized state [1]. In our current treatment, we are able to predict qualitatively the existence of such coherence, and quantitatively the width of the peaks in the distribution which arise due to the imperfect locking of coherent states because of the independent noise added to the system, called modulational intermittency [18], Fig. 2. The qualitative behavior can be seen in the raster plots and phase space snapshots of Fig. 1. It should be noted that in the clustering case, individual oscillators switch stochastically between clusters when their locking to a certain cluster fails due to the independent noise.

The distribution described by Eq. 5 was found by iterating the solution until a stationary condition was reached. It is seen that the results agree with simulation, Fig. 3.

4. Conclusion

We have formulated a method for finding the global distribution function describing coherent states of oscillators receiving common Poisson impulses by using the averaging method. This effectively reduces the problem to a 1 body problem, with a jump-diffusion forward Kolmogorov equation that has an intuitive interpretation. The global outlook is formulated in a less restrictive way than the local stability analysis used earlier [10, 11] because it takes into account the effects of indepedent noise, and may prove useful in treating common-noise induced coherence for more general types of oscillators, not just limit-cycles.

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References

[1] H. Nakao, K. Arai and Y. Kawamura, Phys. Rev. Letters **98**, 184101 (2007).

[2] T. Royama, *Analytical Population Dynamics* (Chapman & Hall, New York, 1992)

[3] R. Roy and K.S. Thornburg, Jr., Phys. Rev. Letters **72**, 2009 (1994).

[4] A. Uchida, R. McAllister, and R. Roy, Phys. Rev. Letters **93**, 244102 (2004).

[5] Z. F. Mainen and T. J. Sejnowski, Science **268**, 1503 (1995).

[6] L. M. Pecora and T. L. Carroll, Phys. Rev. Letters, **64**, 821 (1990).

[7] A. Pikovsky, M. Rosenblum and J. Kurths, *Synchronization: A universal concept in nonlinear sciences* (Cambridge University Press, 2001).

[8] C. Zhou and J. Kurths, Phys. Rev. Letters, **88** 230602 (2002)

[9] J. Teramae and D. Tanaka, Phys. Rev. Letters **93**, 4103 (2004).

[10] K. Nagai and H. Nakao, Phys. Rev. E **71**, 6217 (2005).

[11] H. Nakao, K. Arai and K. Nagai, Phys. Rev. E **72**, 6220 (2005).

[12] K. Yoshimura, P. Davis and A. Uchida, arXiv:0705.4520 (2007).

[13] A. T. Winfree *The Geormetry of Biological Time* (Sprinter, New York, 2001).

[14] D. L. Snyder *Random Point Processes*, (John Wiley & Sons, Inc. 1975).

[15] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences* (Springer, 1997).

[16] F. B. Hanson, *Applied Stochastic Processes and Control for Jump-Diffusions*, SIAM Books, 2007.

[17] Y. Kuramoto, *Chemical Oscillation, Waves, and Turbulence* (Springer-Verlag, Tokyo, 1984) (republished by Dover, New York, 2003).

[18] H. Fujisaka and T. Yamada, Prog. of Theor. Phys., 74, 918 (1985).