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Consistency in driven nonlinear systems

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Abstract – We introduce a viewpoint of consistency in non-autonomous nonlinear dynamical systems. We use the Lorenz model driven by chaos and colored noise signals. Consistent behavior is observed even though the system is in a chaotic motion. The occurrence of consistency can be evaluated by the conditional Lyapunov exponent. We visualize contraction and expansion regions on the driven trajectory in the phase space by using the local conditional Lyapunov exponents. The distribution of the local conditional Lyapunov exponents is plotted when the drive amplitude is increased, and the transition of the emergence of consistency is found. Multiple basins of consistency are also investigated.

1. Introduction

Many nonlinear dynamical systems have a good ability to reproduce consistent response when driven by a repeated external signal. Consistency is defined as the reproducibility of response waveforms in a nonlinear dynamical system driven repeatedly by a signal, starting from different initial conditions of the system [1]. The concept of consistency raises a general question: how does a dynamical system behave with an arbitrary drive signal? Consistency may be a universal concept for non-autonomous dynamical systems.

Consistency of dynamics is essential for information transmission in biological and physiological systems and for reproduction of spatiotemporal patterns in nature. Consistency has been observed experimentally in rat cortical neurons driven by a repeated noise signal [2], and consistency may play an important role for information processing in neuronal activity. For engineering applications, consistency could be useful for the implementation of physical one-way function [3] and for generating common secure keys for stream cipher. Consistency tests could be applied in non-invasive diagnostic procedures to detect changes in system parameters due to aging, catastrophic events or other system changes [4].

Consistency is a generalized concept of chaos synchronization in coupled nonlinear dynamical systems [5]. The conditional stability with respect to a drive signal is essential for the emergence of both consistency and synchronization. However, we do not restrict a type of drive signal for consistency. One can deal with non-autonomous systems driven by any types of external signals in terms of consistency. Many nonlinear dynamical phenomena, such as identical synchronization of chaos [5], generalized synchronization [6], noise-induced synchronization [7], stochastic resonance [8], and coherence resonance [9] may be interpreted as a consistent behavior of non-autonomous systems with respect to external drive signals.

In this study we introduce a viewpoint of consistency in non-autonomous dynamical systems. We use the Lorenz model driven by chaos and colored noise signals. We observe consistency of the driven Lorenz system even though the system is in a chaotic motion. We visualize contraction and expansion regions on the driven chaotic attractors in the phase space by using the local conditional Lyapunov exponents. The distribution of the local conditional Lyapunov exponents is changed when the drive amplitude is increased. This change indicates the emergence of consistency. The basins of consistency for the trajectories starting from different initial conditions are also discussed.

2. Numerical results

2.1. Temporal waveforms and cross correlations

We used the Lorenz model, consisting on a simple set of ordinary differential equations [10]. We used an additive drive signal $s(t)$ with the amplitude of D to the y-variable for a non-autonomous Lorenz model. In our calculation we selected $s(t)$ as a deterministic or a stochastic noise signal: a Lorenz chaos generated from the same set of equations at D = 0 and an exponentially correlated colored noise (Ornstein-Uhlenbeck process) [11].

The temporal waveforms of the drive signal and x-variable starting from two initial conditions are shown in Figs. 1(a) and 1(b). There is a transient before the two time series of x-variable converge into the identical temporal waveform. We calculated the consistency parameter C as a cross correlation [1] of two temporal waveforms starting from two different initial conditions after transient. The consistency parameter C is calculated and plotted as a function of the drive amplitude in the cases of a Lorenz chaos and a colored-noise drive signal, as shown in Figs. 1(c) and 1(d). C increases and reaches to

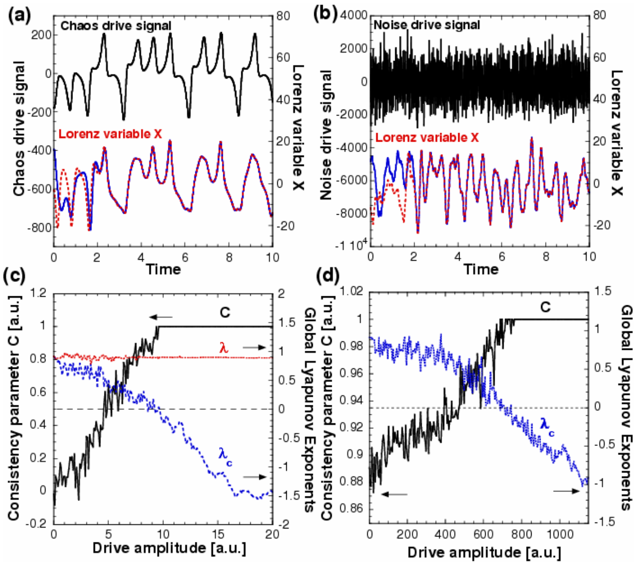

Fig. 1 (a),(b) Temporal waveforms of the drive signal and the *x*-variable of the driven Lorenz model starting from two different initial conditions. (a) Chaos drive and (b) colored noise drive. (c),(d) The cross correlation *C* between two temporal waveforms (black curve), the conditional Lyapunov exponent λ_c (blue curve), and the Lyapunov exponent λ (red curve) as a function of drive amplitude. (c) Chaos drive and (d) colored noise drive. $\sigma = 10$, $r = 28$, and $b = 8/3$ are used for the Lorenz model to observe chaotic behaviors without a drive signal. *y*-variable of the Lorenz model is additively driven by a Lorenz chaos and a colored noise signal.

one as the drive amplitude is increased. We calculated the conditional Lyapunov exponent λ_c and plotted as a function of the drive amplitude in Figs. 1(c) and 1(d). The sign of λ_c changes from positive to negative as the drive amplitude is increased. It is found that negative λ_c corresponds to $C = 1$. We also calculated the (normal) Lyapunov exponent λ only for the chaos-driven Lorenz model (λ cannot be calculated for noise-driven systems because of infinite dimensionality). λ is plotted as a function of the drive amplitude as shown in Fig. 1(c). It is worth noting that λ_c (measure of consistency) is different from λ (measure of chaoticity) in the presence of drive signal. At large drive amplitude consistent behavior can be observed even in a chaotic motion at the condition of $\lambda > 0$ and $\lambda_c < 0$. The measure of chaos and consistency is clearly distinguished in terms of λ and λ_c.

2.2. Local conditional Lyapunov exponent

The calculation of λ_c has been already carried out in the context of generalized synchronization and noise-induced synchronization [6,7]. However, the condition of the emergence of consistency in driven dynamical systems has not been well understood. To investigate deep insight of consistency, we introduce the *local* conditional Lyapunov exponent λ_{lc} as,

$$\lambda_{lc} = \frac{1}{\Delta t} \log \frac{\left\| \xi(t+\Delta t) \right\|}{\left\| \xi(t) \right\|} \qquad (1)$$

where $\xi(t)$ is the linearized variables of the Lorenz system. For numerical calculation Δt has a finite time corresponding to a time step of the numerical calculation. We normalized $\xi(t)$ at each step of numerical integration so that the vector $\xi(t)$ can maintain a unit vector.

We plotted the local conditional Lyapunov exponent λ_{lc} on the trajectory of the Lorenz butterfly attractor in the phase space. Figure 2(a) shows λ_{lc} plotted on the trajectories in the *x-z* plane of the three dimensional phase space at $y = 0$ without a drive signal. The expansion ($\lambda_{lc} > 0$) and contraction ($\lambda_{lc} < 0$) regions are observed and are well separated on the attractor. The upper part of the attractor along the *z* axis mainly has contraction regions, whereas the lower part of the attractor corresponds to expansion regions. Figures 2(b) and 2(c) show λ_{lc} on the trajectories when the *y* variable of the Lorenz model is additively driven by a Lorenz chaos and a colored noise signal, respectively. For chaos drive the attractor becomes larger than the original (non-driven) attractor, however, λ_{lc} is distributed similarly to Fig. 2(a). Contraction regions exist on the upper part of the attractor and expansion regions appear on the lower part, although the contraction and expansion regions do not exactly match to those in Fig. 2(a). For noise drive, the distribution of λ_{lc} on the attractor still preserves, even though the attractor becomes more irregular.

To understand the emergence of consistency, we calculated the probability distribution of λ_{lc} when the drive amplitude is increased. Figure 3(a) shows the three-dimensional picture of the probability distribution of λ_{lc} when the noise-drive amplitude is continuously changed. The distribution of λ_{lc} changes and becomes smooth as the drive amplitude is increased. Figures 3(b)-3(d) show the probability distribution of λ_{lc} at a constant drive amplitude. Without the drive signal there are two peaks of the distribution: a large sharp peak at the negative λ_{lc} and a small broadened peak at the positive λ_{lc} (Fig. 3(b)). The broadened peak at the positive λ_{lc} decreases as the drive amplitude is increased, and finally disappears at large drive amplitudes as shown in Fig. 3(d). The shape peak at the negative λ_{lc} remains at the same value. The positive components of λ_{lc} shifts to negative parts as the drive amplitude increases, and consistency appears at large drive amplitudes. Note that the distribution of λ_{lc} does not change drastically and the slight change of λ_{lc} by the drive signal results in the emergence of consistency. The conditional Lyapunov exponent λ_c corresponds to the average of the distribution of λ_{lc}. The transition of λ_{lc} from

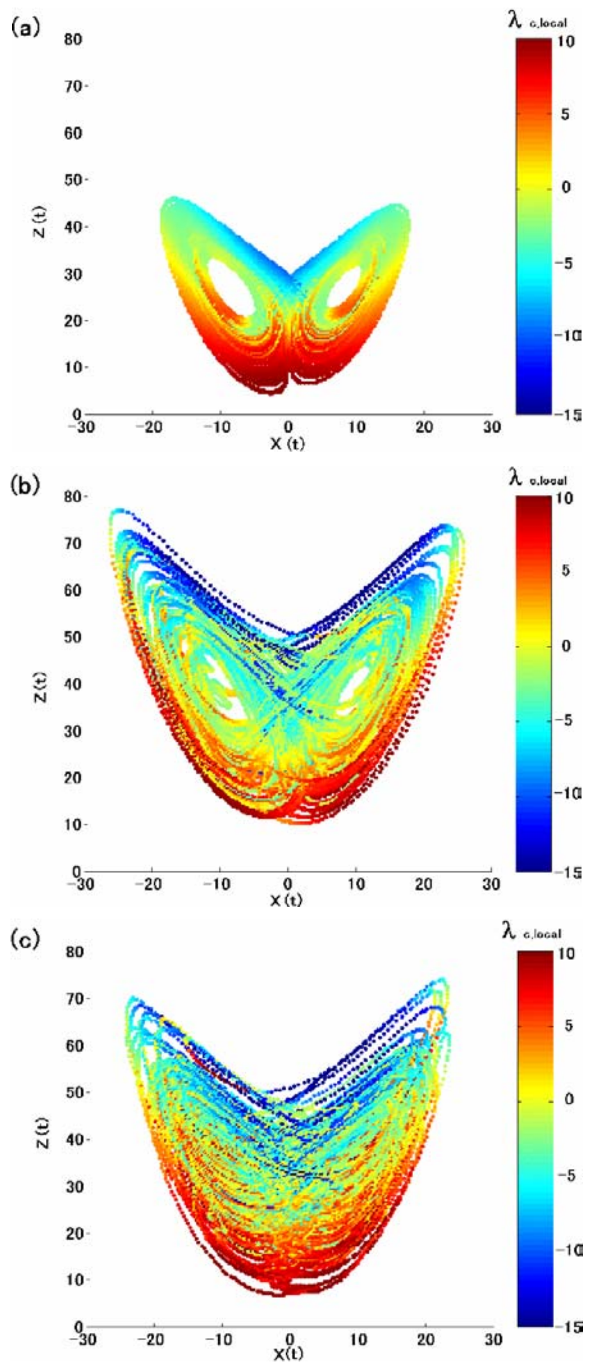

Fig. 2 The driven butterfly. (a) No drive, (b) chaos drive D = 15 , and (c) colored noise drive D = 40. The color indicates the local conditional Lyapunov exponent λ_{lc} plotted on the trajectory in the x-z plane of the phase space. Expansion (red) and contraction (blue) regions are observed on the trajectory.

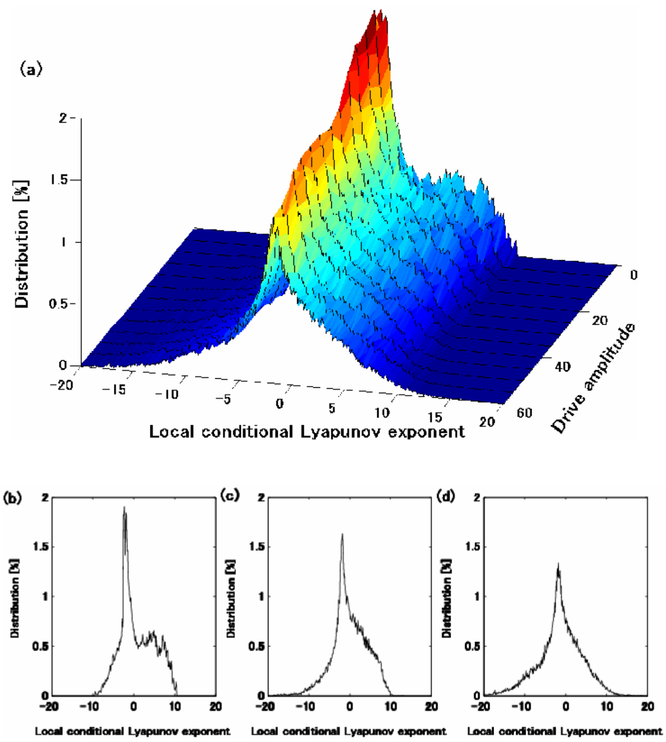

Fig. 3 (a) Three dimensional picture of the distribution of local conditional Lyapunov exponent λ_{lc} as a function of drive amplitude. (b)-(d) Distribution of λ_{lc} at constant drive amplitudes. (b) D = 0, (c) D = 30, and (d) D = 60. The conditional Lyapunov exponent λ_c is obtained from the average of the distribution of λ_{lc}: (b) λ_c = 0.863, (c) λ_c = - 0.058, and (d) λ_c = - 1.322.

positive to negative components shown in Fig. 3 indicates the emergence of consistency. λ_{lc} is thus a good measure to visualize the contraction and expansion regions in the phase space (Fig. 2) and to evaluate the transition of the emergence of consistency (Fig. 3).

2.3. Multiple basins of consistency

The basin of consistency (regions of initial conditions that can lead to consistent trajectory) is a very important issue. We calculated the basin of consistency in the phase space of the noise-driven Lorenz model. Here we used r = 13 so that the Lorenz system has two fixed points without a drive signal. x-variable is additively driven by a colored-noise signal. In the case of small drive amplitude of D = 10 we obtained two types of consistent trajectories depending on the initial conditions. We calculated the basins of consistency by changing the initial conditions in the phase space. We used a set of colored-noise drive signals (ten different temporal waveforms) and calculated the probability to converge into one of the two consistent trajectories. The probability is plotted as a color code on

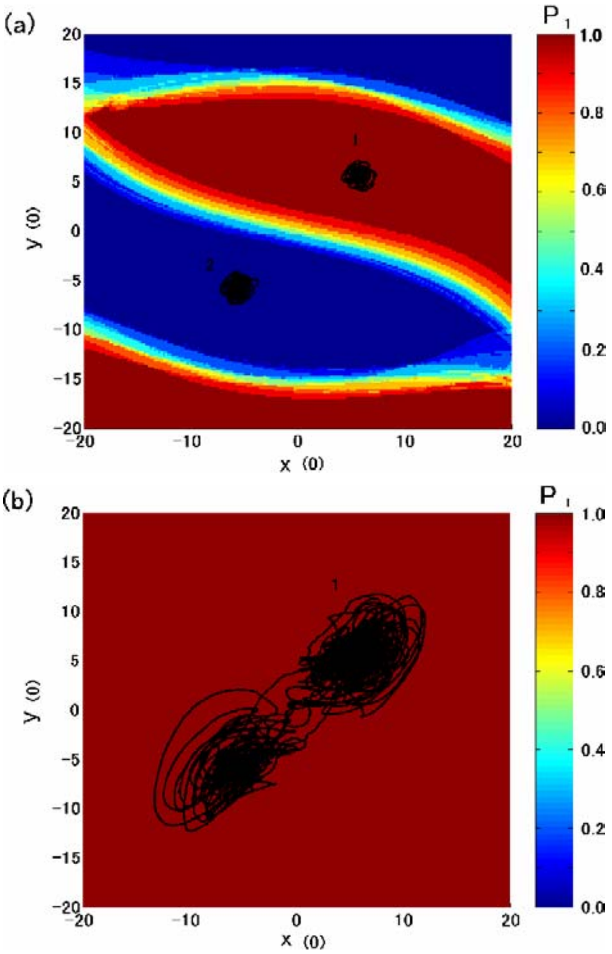

Fig. 4 (a) Multiple basins of consistency at D = 10. Black lines indicate two consistent trajectories. Color indicates the probability of converging into one of the two trajectories (trajectory 1). Red color corresponds to the basin for the trajectory 1, and blue color corresponds to the basin for the trajectory 2. (b) Single basin of consistency at D = 30. Black line indicates the consistent trajectory 1. Red color indicates the basin for the trajectory 1. The Lorenz model has two fixed points without a drive signal (r = 13). x-variable of the Lorenz model is additively driven by a colored-noise signal.

the x-y plane of the three-dimensional phase space at z = 12, as shown in Fig. 4(a). Two trajectories (black parts) are located near the original fixed points and there exists two basins of consistency (red and blue regions in Fig. 4(a)) for the two trajectories. It is worth noting that the trajectory is exactly identical after transient when starting from the same basin. The existence of more than one basin is called *multiple basins of consistency*. The boundary of the two basins is not critical and there is some chance of escaping to the other trajectory around the basin boundary. As the drive amplitude is increased at D = 30 the two trajectories become larger and merge together. There exists only one trajectory and single basin of consistency for all the initial conditions, as shown in Fig. 4(b). It is found that the size of basin depends on the amplitude of the drive signal.

3. Conclusion

We have introduced a viewpoint of consistency in non-autonomous dynamical systems. We have used the Lorenz model driven by chaos and colored noise signals. Consistency is observed even though the system is in a chaotic motion. We have visualized contraction and expansion regions on the driven chaotic attractors in the phase space by using the local conditional Lyapunov exponents λ_{lc}. The distribution of λ_{lc} is changed when the drive amplitude is increased. The emergence of consistency is clearly observed by using the distribution of λ_{lc}. Multiple basins of consistency have been also found. These aspects of consistency may be general features that can be observed in many non-autonomous nonlinear systems. Consistency may provide a new viewpoint in non-autonomous nonlinear classical and quantum systems.

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