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A stability condition for a simple type of two-dimensional Discrete-time Binary Cellular Neural Networks

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Abstract—We give a stability condition for a two-dimensional discrete-time binary cellular neural networks with an A-template having a simple type of symmetric connection characterized by two independent parameters α and β. The results are extension of our previous one-dimensional cases [10],[11].

1. Introduction

One of the most fundamental problems of *two-dimensional cellular neural networks*(CNN) is the stability problem and it has long been investigated from the theoretical and practical points of view [1]-[4]. But the problem is not completely solved even for binary cellular neural networks[5]-[7].

Concerning the stability conditions of a *1-dimensional discrete-time binary cellular neural network*, Sato et al.[7] gave the necessary and sufficient conditions in terms of changeable sets and Nishi et al.[8], [10], [11] and Hara et al.[9] recently gave the necessary and sufficient conditions in terms of system parameters for both no input case and nonzero input case. On the other hand Thiran et al[6] studied on 1-dimensional *analog* CNNs and gave the stability conditions for them. So the stability problem for a 1-dimensional systems with one-neighborhood connection is considered to be solved .

In this paper we study on the stability for a 2-dimensional discrete-time binary cellular networks (abbreviated as 2-D DBCNN) with an A-template having a simple type of *symmetric connection* characterized by two independent parameters α and β (see Eq.(1) below). We will give a necessary and sufficient condition for this type of the 2D DBCNNs to be stable. We study only no input case. The method used is resemble to the one used in the one-dimensional cases[8]–[9].

2. Preliminaries

Let $x(k) = [x_{ij}(k)]$ be an $n \times n$ binary state matrix of a 2-D DBCNN S at time k. The behavior of a 2-D DBCNN we consider in this paper can generally be described by the equation:

$$x_{ij}(k+1) = \text{sgn}\,[\alpha x_{i,j}(k) + \beta x_{i-1,j}(k) + \beta x_{i+1,j}(k) + \beta x_{i,j-1}(k) + \beta x_{i,j+1}(k) + \theta] \quad (1)$$
$$(i,j = 1,2,\cdots,n;\ k = 0,1,2,\cdots)$$

where θ is the threshold value. Thus the cell (i,j) has a self-feedback with the magnitude α and has connections only with neighboring four states $(i-1,j)$, $(i+1,j)$, $(i,j-1)$ and $(i,j+1)$ where the connection coefficients are the same value, β [1]. In particular $x(0)$ is the initial state matrix, which can be used as another input data in many applications.

When we calculate $x_{ij}(k+1)$ by Eq.(1), we have to define the boundary values $x_{0,j}(k)$, $x_{n+1,j}(k)$, $x_{i,0}(k)$, and $x_{i,n+1}(k)$ for the state matrix. In this paper we assume the fixed boundary, which means that $x_{0,j}(k)$, $x_{n+1,j}(k)$, $x_{i,0}(k)$ and $x_{i,n+1}(k)$ are binary constants independent of k.

Definition 1: A 2-D DBCNN S is said to be stable, if no limit cycle occur for any $x(0)$, any boundary conditions on x, and any n. A 2-D DBCNN being not stable are said to be unstable.

Based on the Definition 1, we consider the problem:
Problem: Prescribed coefficients α, β and θ, is the system S stable or not?
In this paper we give the answer to above problem.

3. Main results on the stability

We first describe the main results in this paper.

[1]In the case of cellular automata the "sgn" function in Eq.(1) should be replaced with an arbitrary logic function of $x(k)$.

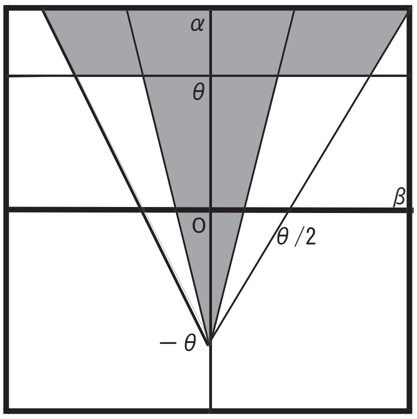

Figure 1: Stable and unstable regions

Theorem 1: The system S described by Eq.(1) is stable if and only if one of the following conditions i) and ii) holds:
i) $\alpha > 4|\beta| - \theta$
ii) $\alpha > 2|\beta| - \theta$ and $\alpha > \theta$

The stable regions are shown by the shaded area in Fig. 1, where the vetrical axis means α-axis and the horizontal axis means the β-axis. On the other hand the white area in Fig. 1shows unstable region.

The proof of Theorem 1 is done by proving the following two theorems.

Theorem 2: The system S described by Eq.(1) is stable if one of the following conditions i) and ii) in Theorem 1 holds.
i) $\alpha > 4|\beta| - \theta$
ii) $\alpha > 2|\beta| - \theta$ and $\alpha > \theta$

Theorem 3: The system S described by Eq.(1) is unstable if one of the following conditions iii) and iv) in Theorem 1 holds.
iii) $\alpha < 2|\beta| - \theta$
iv) $\alpha < 4|\beta| - \theta$ and $\alpha < \theta$

The followings are the proofs of Theorems 2 and 3. The proof is basically similar to but is more complicated than those in [8], [9]

4. Proof of theorems

In the following proof we assume that $\beta \geq 0$ but the case of $\beta < 0$ can be treated quite similarly. So we write $|\beta|$ instead of β so that the result is valid for the case of $\beta < 0$. We also assume that $\theta \geq 0$ without loss of generality.

4.1. Changeable pattern and invariant pattern

The 5-tuple ($\phi \equiv$) ($x_{i,j}(k)$, $x_{i-1,j}(k)$, $x_{i+1,j}(k)$, $x_{i,j-1}(k)$, $x_{i,j+1}(k)$) takes one of 32 patterns such as $(-,-,-,-,-)$, $(-,-,-,-,+)$ and $(+,+,+,+,+)$, where "+" and "−" mean +1 and −1, respectively. Assume that the parameters α, β and θ are given. Then for some of these 5-tuples $x_{ij}(k+1)$ changes from $x_{ij}(k)$ and for other 5-tuples $x_{ij}(k+1)$ remains the same as $x_{ij}(k)$.

Definition 2: We call the former 5-tuples "*changeable patterns*" and the latter ones "*invariant patterns*".

Due to the symmetrical structure of CNN we consider, it depends only on the value (+1 or −1) of the center cell (i,j) and on the number of positive (or negative) value of four adjacent cells, $(i-1,j)$, $(i+1,j)$, $(i,j-1)$ and $(i,j+j)$, Let Np(resp., Nn; $Np+Nn=4$) denote the number of positive (resp., negative) adjacent cells.

4.2. α- and $\bar{\alpha}$-terms

We see from Eq.(1) that the behavior of the system can be determined only by the values of $\pm\alpha \pm \beta \pm \beta \pm \beta \pm \beta + \theta$.

Definition 3: We call the terms $+\alpha\pm\beta\pm\beta\pm\beta\pm\beta+\theta$ *α-terms* and $-\alpha\pm\beta\pm\beta\pm\beta\pm\beta+\theta$ *$\bar{\alpha}$-terms* respectively.

4.3. Conditions A-i and $\bar{A}$-j

We define the following six conditions for the α-terms and also six for $\bar{\alpha}$-terms as follows:

Condition A-5: $+\alpha + 4|\beta| + \theta < 0$
Condition A-4: $+\alpha + 2|\beta| + \theta < 0 < +\alpha + 4|\beta| + \theta$
Condition A-3: $+\alpha + \theta < 0 < +\alpha + 2|\beta| + \theta$
Condition A-2: $+\alpha - 2|\beta| + \theta < 0 < +\alpha + \theta$
Condition A-1: $+\alpha - 4|\beta| + \theta < 0 < +\alpha - 2|\beta| + \theta$
Condition A-0: $0 < +\alpha - 4|\beta| + \theta$

Similarly Conditions $\bar{A}$-j ($j = 0, 1, \cdots, 5$) are defined as follows:

Condition $\bar{A}$-5: $-\alpha + 4|\beta| + \theta < 0$
Condition $\bar{A}$-4: $-\alpha + 2|\beta| + \theta < 0 < -\alpha + 4|\beta| + \theta$
Condition $\bar{A}$-3: $-\alpha + \theta < 0 < -\alpha + 2|\beta| + \theta$
Condition $\bar{A}$-2: $-\alpha - 2|\beta| + \theta < 0 < -\alpha + \theta$
Condition $\bar{A}$-1: $-\alpha - 4|\beta| + \theta < 0 < -\alpha - 2|\beta| + \theta$
Condition $\bar{A}$-0: $0 < -\alpha + 4|\beta| + \theta$

These conditions are restated as follows: Assume that $x_{ij}(k) = 1$. Then

(a.1) Condition A-0 means that $x_{ij}(k+1) = 1$ independent of Np, which means that there is no changeable pattern of the type (+, *, *, *, *),

(a.2) Condition A-i ($i = 1, \cdots, 4$) means that $x_{ij}(k+1) = 1$ if $Np \geq i$ and $x_{ij}(k+1) = -1$ if $Np < i$, i.e., $(+,*,*,*,*)$ is a changeable pattern if $Np < i$,

(a.3) Condition A-5 means that $x_{ij}(k+1) = -1$ independent of Np, which means all 5-tuples of the type (+, *, *, *, *) are changeable patterns.

Similarly assume that $x_{ij}(k) = -1$. Then

(b.1) Condition $\bar{A}$-0 means that $x_{ij}(k+1) = 1$ independent of Np.

(b.2) Condition $\bar{A}$-j ($j = 1, \cdots, 4$) means that $x_{ij}(k+1) = 1$ if $Np \geq j$ and $x_{ij}(k+1) = -1$ if $Np < j$, and (b.3) Condition $\bar{A}$-5 means that $x_{ij}(k+1) = -1$ independent of Np.

Examples of patterns for each condition are illustrated in Figs. 2 and 3.

A-5

	+	
+	+	+
	+	

A-4

	+	
−	+	+
	+	

A-3

	−	
−	+	+
	+	

A-2

	−	
−	+	−
	+	

A-1

	−	
−	+	−
	−	

A-0

Figure 2 α-terms

$\bar{A}$-5

	+	
+	−	+
	+	

$\bar{A}$-4

	+	
−	−	+
	+	

$\bar{A}$-3

	−	
−	−	+
	+	

$\bar{A}$-2

	−	
−	−	−
	+	

$\bar{A}$-1

	−	
−	−	−
	−	

$\bar{A}$-0

Figure 3 $\bar{\alpha}$-terms

5. Investigation of 36 combinations

Previously we defined Conditions A-i $(i = 0, 1, \ldots, 5)$ for α-terms and Conditions $\bar{A}$-$j (j = 0, 1, \ldots, 5)$ for $\bar{\alpha}$-terms. To investigate the stability conditions of the system S it is sufficient to investigate *all 36 combinations* of Conditions A-i and Conditions $\bar{A}$-j $(i, j = 0, 1, \ldots, 5)$. We indeed examined these 36 combinations in detail and verified which combination is stable. But to write this process is very tedious. In the following we investigate heuristically only four cases, from which we can derive the necessary and sufficient conditions for the stability.

Before that we should note the following:

Remark 1: If a 5-tuple ϕ is a changeable pattern on Condition A-i_0, then it is also a changeable pattern on Conditions A-i $(i \leq i_0)$. Similarly if a 5-tuple ϕ is a changeable pattern on Condition $\bar{A}$-j_0, then it is also a changeable pattern on Conditions $\bar{A}$-j $(j \geq j_0)$.

5.1. Stable regions

We first consider:

Case 1: $\alpha > 4|\beta| - \theta$

We see that Condition A-0 holds in this case. Therefore any α-term is invariant pattern (See Fig. 2). In other words if $x_{ij}(k) = 1$, then $x_{ij}(k+1) = 1$.

Even if $x_{ij}(k)$ changes from -1 to 1 (See Fig.3), we see that any limit cycle does not happen in this case.

Case 2: $\alpha > 2|\beta| - \theta$ and $\alpha > \theta$

These inequalities show that Conditions A-1 holds in α-terms, Condition $\bar{A}$-3 holds in $\bar{\alpha}$-terms.

We verified which combination is stable from all 36 combinations carefully. As we cannot write all cases for the lack of space , we will state only key points briefly.

We see that changeable patterns are only a positive isolated point (See Fig.4) in α-terms.

Similarly there are a negative isolated point (Fig.5) and a negative end point(Fig.6) in $\bar{\alpha}$-terms.

In the following 'positive'(resp., 'negative') is abbreviated to 'p-'(resp.,'n-').

	−	
−	+	−
	−	

Figure 4 p-isolated point

	+	
+	−	+
	+	

Figure 5 n-isolated point

	+	
−	−	+
	+	

Figure 6 n-end point

	−	
−	−	+
	+	

Figure 7 n-corner point

When p-isolated points and n-isolated points are arranged checkerwise, each point may change between +1 and -1 alternatively. But the System can not have Limit Cycles in fixed boundary conditions in which the boundary values are constant +1 or -1.

Considering the size of 2×2 as shown Fig.8, each isolated point reverses to opposite state at each corner. The point that became new state never returns i.e. stable, because n-isolated point (resp., p-isolated point) becomes p-point (resp., n-point) which has $N_p \geq 1$ (resp., $N_n \geq 1$). That is, new point is not isolated point.

Supposing the case of bigger size, the states of cells become fixed gradually from periferal cells, and converged finaly.

	−	+	
−	+	−	+
+	−	+	−
	+	−	

(a)

	−	+	
−	−	+	+
+	+	−	−
	+	−	

(b)

Figure 8 Example 1

From the above we conclude that Theorem 2 in Section 3 holds.

5.2. Unstable regions

We next consider:

Case 3: Region $\alpha < 2|\beta| - \theta$

This inequality shows that Condition A-2 in the α-terms (See Fig.2) and $\bar{A}$-3 in the $\bar{\alpha}$-terms (See Fig.3).

The changeable patterns are p-isolated point (Fig.4) and p-end point in the α-terms, on the otherhand n-isolated point (Fig.5) and n-end point (Fig.6) in the $\bar{\alpha}$-terms.

Considering the size of 2×2 as shown Fig.9(a), each end point changes states at each corner. The end points that became new states changes in turn between (a) and (b) of Fig.9.

	−	+	
+	−	+	−
−	+	−	+
	+	−	

(a)

	−	+	
+	+	−	−
−	−	+	+
	+	−	

(b)

Figure 9 Example 2

Case 4: Region $\alpha < 4|\beta| - \theta$ and $\alpha < \theta$
This inequalities shows that A-1 holds in the α-terms and $\bar{A}$-2 holds in the $\bar{\alpha}$-terms.

Therfore we see the changeable pattern is only p-isolated point (See Fig.4) in the α-terms. In the $\bar{\alpha}$-terms, changeable set has three changeable patterns(See Fig.5,6,7).

Considering the size of 2×2 as shown Fig.10, each n-corner point changes to a p-isolated point at each corner, and each p-isolated point changes to an n-corner point.

On the whole pattern, inner 4 points change in turn between (a) and (b) of Fig.10.

	−	−	
−	−	+	−
−	+	−	−
	−	−	

(a)

	−	−	
−	+	−	−
−	−	+	−
	−	−	

(b)

Fig.10 Example 3

From the above we conclude that Theorem 3 in Section 3 holds.

6. Conclusion

We gave the necessary and sufficient conditions for the stability of 2-D DBCNN without input.

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