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An Influence of Second Harmonic Excitation on Rotation in Parametric Pendulum

Yuichi Yokoi†, Tsuyoshi Higuchi†, and Takashi Hikihara‡

†Division of Electrical Engineering and Computer Science, Nagasaki University
1-14 Bunkyo, Nagasaki 852-8521, Japan
‡Department of Electrical Engineering, Kyoto University
Katsura, Nishikyo, Kyoto 615-8510, Japan
Email: {yyokoi,thiguchi}@nagasaki-u.ac.jp, hikihara.takashi.2n@kyoto-u.ac.jp

Abstract—A double frequency vibration including the second harmonic is introduced as a vertical excitation in a parametrically excited pendulum. The harmonic arises in the motion of a body, corresponding to the excitation, which is affected by rotation of the pendulum. This paper examines the influence of the harmonic on dynamics associated with periodic rotation. It is clarified that the second harmonic with a certain range of phase facilitates the bifurcation with respect to the amplitude of the fundamental in the excitation.

1. Introduction

Pendulum exchanges a vertical vibration into rotation parametrically. The characteristics have a potential for the application of energy extraction from vibration in nature such as sea wave. The parametric pendulum [1, 2] represents a mechanical pendulum with its pivot point excited vertically. The vertical excitation, called the parametric excitation, is assumed to be a sinusoidal vibration. There appear some steady states in the system. In particular, periodic rotation corresponds to the converting motion suitable for the energy extraction. Analytical solutions are obtained for some simple types of periodic rotations [3, 4] and the domain of existence for rotation is numerically clarified in the excitation parameter plane [2, 5]. These findings are expected to contribute the implementation of the energy extraction.

The parametric excitation is adjusted from the point of view of application. In the extraction of wave energy, horizontal vibration is introduced into the vertical excitation to consider the elliptical motion of sea wave [6]. For the elliptical excitation the characteristics of periodic rotation depends on its rotational direction. The optimum waveform of the parametric excitation is explored for the maximum amount of energy conversion in a pendulum with a motion control [7]. Adjusted excitations cause substantial change to the dynamical property from the sinusoidal excitation.

In this paper, second harmonic is introduced to a sinusoidal excitation. The second harmonic is caused by the motion of a body that exerts the parametric excitation on the pivot point of a pendulum. In general, the motion is fixed at a sinusoidal vibration in the parametric pendulum on the assumption that the motion of the pendulum exerts little effect on the body. This implies that the pendulum extracts little energy in comparison with the vibration energy of the body. The ratio of extracted energy is associated with the size of the effect. For large ratio of energy extraction, the fundamental and the second harmonic are dominant in the motion of the body. This study clarifies that the second harmonic with a certain range of phase facilitates the bifurcation with respect to the amplitude of the fundamental.

2. Second Harmonic Excitation

A model is derived for the parametric pendulum with a second harmonic excitation. Rotation of a pendulum affects the motion of its supporting rig in a pendulum rig system under a sinusoidal vibration. For periodic rotation, the second harmonic arises in the motion of the rig corresponding to the parametric excitation.

2.1. Pendulum Rig System

Figure 1 shows a pendulum rig system. The system is constructed of a pendulum with the mass m and the length l, and a rig with the mass M which supports the pendulum. It is assumed that the motion of the supporting rig is constrained in the vertical direction. The vertical displacement x determines the position of the rig. The position of the pendulum is defined with the angular displacement θ from the downward position. The behavior of the pendulum rig system is represented by the following equations of motion:

$$ml^2\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + c\frac{\mathrm{d}\theta}{\mathrm{d}t} + ml\left(g + \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right)\sin\theta = 0, \tag{1a}$$

$$(M+m)\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + (M+m)g = -ml\left[\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\sin\theta + \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\cos\theta\right] + F(t), \tag{1b}$$

where g is the gravity acceleration, c depicts the viscous damping coefficient, and force applied to the supporting rig is expressed as

$$F(t) = (M+m)g + A\cos\Omega t, \tag{2}$$

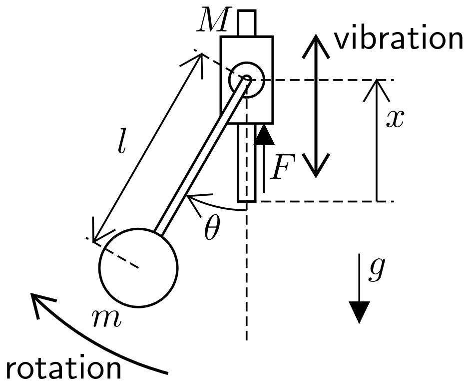


Figure 1: Schematic of a mechanical pendulum rig system.

where A and Ω stand for the amplitude and the angular frequency of the vibration, respectively. Eq. (1b) describes that the motion of the pendulum exerts a force on the rig. The force increases with the mass ratio of the pendulum to the whole system.

2.2. Parametric Excitation with Second Harmonic

The parametric pendulum with the second harmonic excitation is derived by restricting the motion of the pendulum to periodic rotation. Substituting (1b) and (2) into (1a) and non-dimensionalizing it with respect to $\Omega_0 = \sqrt{g/l}$, the motion of the pendulum is described by

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t'^2} = \frac{1}{1-\mu\sin^2\theta}\Bigg[-c'\frac{\mathrm{d}\theta}{\mathrm{d}t'} - \left\{1 + A'\cos\Omega' t' - \mu\left(\frac{\mathrm{d}\theta}{\mathrm{d}t'}\right)^2\cos\theta\right\}\sin\theta\Bigg]. \tag{3}$$

In the equation, $t' = \Omega_0 t$ depicts the dimensionless time, $\mu = m/(M+m)$ the ratio of mass, $c' = c/ml^2\Omega_0$ the dimensionless damping coefficient, and $A' = A/(M+m)g$ and $\Omega' = \Omega/\Omega_0$ are the dimensionless amplitude and angular frequency of the excitation. Here it is noted that the range of the mass ratio is $0 \leq \mu \leq 1$.

Now we consider that c', μ, and A' are sufficiently small. Then the approximation $1/(1-\mu\sin^2\theta) \approx 1 + \mu\sin^2\theta$ is available. Because the motion of the pendulum is restricted to periodic rotation, the variable $\theta(t')$ can be approximated as $\theta(t') \approx \Omega' t' + \theta_0$, where θ_0 denotes the initial value of $\theta(t')$. By substituting the above approximations into (3) and neglecting the second order of the above small parameters, the following equations are obtained for the parametric pendulum with the second harmonic excitation:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = v, \tag{4a}$$

$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = -\gamma v - [1 + p\cos\omega\tau + \alpha\cos(2\omega\tau+\phi)]\sin\theta. \tag{4b}$$

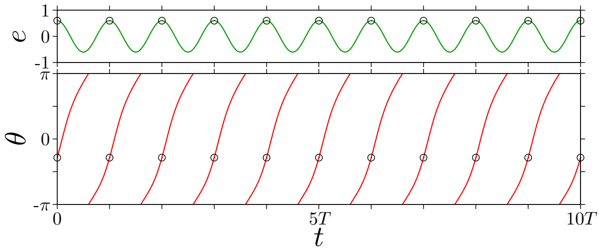
This equation is normalized with $\tau = \sqrt{1+\mu/2}(t' + \beta/\Omega')$ and the other parameters are

$$\beta = \arctan\left(-\frac{\mu\Omega'^2\sin\theta_0}{A' - \mu\Omega'^2\cos\theta_0}\right), \quad \gamma = c'/\sqrt{1+\mu/2}$$

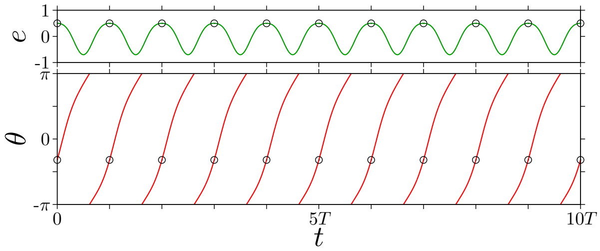
$$p = \sqrt{\frac{(A' - \mu\Omega'^2\cos\theta_0)^2 + (\mu\Omega'^2\sin\theta_0)^2}{1+\mu/2}},$$

$$\omega = \Omega'/\sqrt{1+\mu/2}, \quad \gamma = c'/\sqrt{1+\mu/2},$$

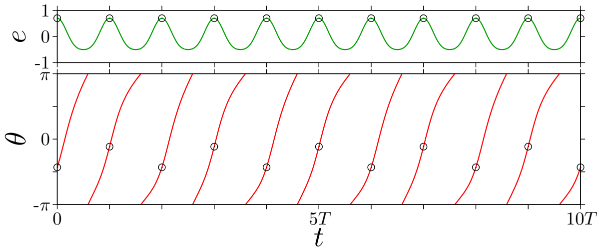
$$\alpha = \mu/(2+\mu), \quad \text{and} \quad \phi = \pi - 2\beta + 2\theta_0.$$



(a) Rotation (1,1) at $\alpha = 0$.



(b) Rotation (1,1) at $\alpha = 0.1$ and $\phi = -\pi$.



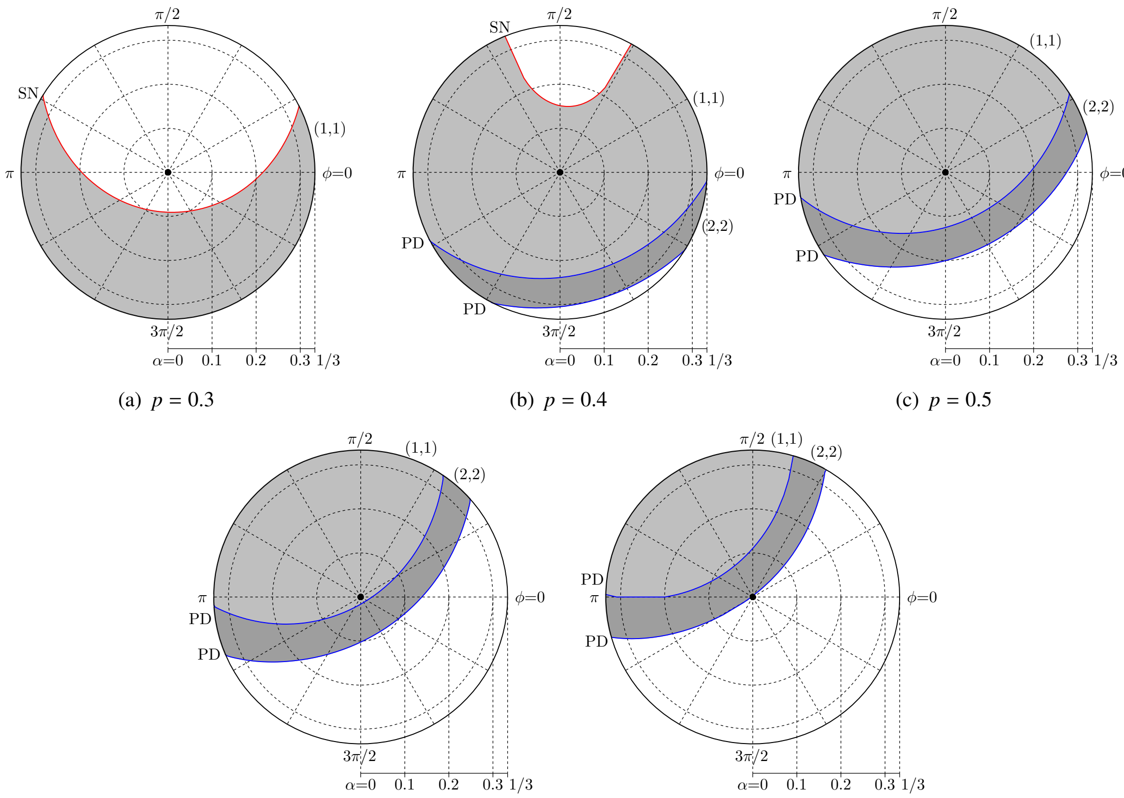
(c) Rotation (2,2) at $\alpha = 0.1$ and $\phi = 0$.

Figure 2: Behaviors for different parameters of the second harmonic excitation at $\gamma = 0.1$, $p = 0.6$, and $\omega = 1.5$. The upper figures show the parametric excitation $e(t)$ and the lower the motion $\theta(t)$ of the pendulum. The circles on the lines indicate the stroboscopic points taken at every excitation period.

In Eq. (4) the parametric excitation can be regarded as double frequency excitation that consists of the fundamental and the second harmonic vibrations. Here the amplitude of second harmonic is restricted in $0 \leq \alpha \leq 1/3$.

3. Numerical Study

An influence of the second harmonic excitation is investigated on the dynamics associated with periodic rotation



(a) $p = 0.3$ (b) $p = 0.4$ (c) $p = 0.5$

(d) $p = 0.6$ (e) $p = 0.7$

Figure 3: Bifurcation diagrams of periodic rotation in (α, ϕ)-plane for some values of p at $\gamma = 0.1$ and $\omega = 1.5$. The light gray corresponds to the domain of existence for rotation $(1, 1)$ and the dark gray the domain for rotation (2,2). SN and PD indicate the boundaries on which the saddle-node and the period-doubling bifurcations take place.

in the parametric pendulum. In this section, periodic rotations are classified based on the expression defined in [5]. A combination of a natural number n and a nonzero integer r corresponds to a periodic rotation described by $\theta(t) = \theta(t - nT) + 2\pi r$, where T denotes the period of the parametric excitation, that is, $T = 2\pi/\omega$. The motion is expressed as rotation (n, r). Throughout the numerical study, the damping parameter is fixed at $\gamma = 0.1$ in Eq. (4) according to the previous works [2, 5, 6].

Figure 2 shows the behaviors for different parameters of the second harmonic in the parametric excitation. The fundamental is regulated at $p = 0.6$ and $\omega = 1.5$ so that a rotation (1,1) appears without the second harmonic. Fig. 2(a) shows the behavior under the single frequency excitation at $\alpha = 0$. Two behaviors excited by the second harmonic are shown in Figs. 2(b) and 2(c). Comparison of these figures reveals an influence of the second harmonic on periodic rotation. Fig. 2(c) shows a different rotation from the others. The second harmonic at $\alpha = 0.1$ and $\phi = 0$ doubles the period of rotation. In contrast, the period does not change at $\alpha = 0.1$ and $\phi = \pi$. These behaviors identify the influence of the second harmonic on periodic rotation. The influence strongly depends on the phase ϕ of the second harmonic.

Figure 3 shows the bifurcation diagrams in (α, ϕ)-plane of the second harmonic for some values of p at $\omega = 1.5$. Each diagram is expressed in polar coordinate of (α, ϕ) and its center point indicates a condition without the second harmonic. The light gray corresponds to the domain of existence for rotation $(1, 1)$ and the dark gray the domain for rotation $(2, 2)$. The center point is included in the gray in each of Figs. 3(b), 3(c), and 3(d). This suggests that the pendulum rotates under a single frequency excitation at each value of p. On the other hand, periodic rotations do not appear in Figs. 3(a) and 3(e).

In the diagram at $p = 0.5$ shown in Fig. 3(c), the light gray contains the circle with radius $\alpha = 0.1$. For any phase ϕ of the second harmonic at the amplitude α, the pendulum sustains its rotation. But the light gray does not cover over the circle with radius $\alpha = 0.2$. The circle has intersections with each of the dark gray and the white. The boundaries of the intersections show that the period-doubling bifurcation arises. Increment of the radius α extends the two intersections. Therefore, the second harmonic with large amplitude can disturb rotation $(1, 1)$ through the period-doubling bifurcation. The disturbance is particularly pronounced for $\phi = 5\pi/3$. Fig. 3(d) shows a bifurcation diagram similar to Fig. 3(c). For the increase of p, the gray shifts to the direction of $\phi = 2\pi/3$, that is, the disturbance increases. Furthermore the influence can also be observed at the condition of $p = 0.7$, shown in Fig. 3(e), so that periodic rotation does not appear without the second harmonic. Thus the second harmonic with phase near to $\phi = 2\pi/3$ sustains periodic rotation.

Next, the diagram at $p = 0.4$ shown in Fig. 3(b) is considered. The gray shifts to the direction of $\phi = 5\pi/3$ from Fig. 3(c). This is consistent with the above discussion. The light gray is surrounded by two types of boundaries. One is the period-doubling bifurcation adjacent to the dark gray, and the other the saddle-node bifurcation neighboring to the white. The saddle-node bifurcation boundary appears because of the decrease of p. The boundary is also observed at the smaller amplitude p in Fig. 3(a). Therefore, the second harmonic suppresses periodic rotation through the saddle-node bifurcation for the smaller value of p. The suppression is enhanced in particular for $\phi = \pi/2$. The influence associated with the saddle-node bifurcation is distinctly confirmed at the condition so that periodic rotation does not appear under the single frequency excitation. The second harmonic can contribute for sustaining the rotation.

The bifurcation diagrams clarify the disturbance of periodic rotation by the second harmonic in the parametric excitation. The dynamics associated with rotation is non-uniformly disturbed for the phase ϕ of the second harmonic. The phase of $\phi = 5\pi/3$ increases the disturbance through the period-doubling bifurcation and that of $\phi = \pi/2$ does the disturbance through the saddle-node bifurcation. The positional relationship between the center point and the bifurcation boundaries in each diagram for the value of p implies that the large effect of the excitation to swing the pendulum up induces the period-doubling bifurcation and the small effect does the saddle-node bifurcation. The size of effect depends on the absolute value of excitation at about $\theta = \pi/2$. In fact, the second harmonic increases the absolute value at $\phi = 5\pi/3$ and decrease it at $\pi/2$.

4. Concluding Remarks

In this paper, we introduce a double frequency vibration including the second harmonic as a parametric excitation applied to a pendulum and examine the influence on the dynamics associated with periodic rotation. The second harmonic excitation results from the consideration of the motion of a body corresponding to the parametric excitation. Numerical study is carried out in order to show the domain of existence for periodic rotation in the parameter plane of the second harmonic. The second harmonic with a certain range of phase facilitates the bifurcation with respect to the amplitude of the fundamental. The understanding of the influence is significant from the point of view of the application, because the domain corresponds to the condition of the parametric excitation for sustaining the motion converting energy.

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References

[1] J. B. McLaughlin, "Period-doubling bifurcations and chaotic motion for a parametrically forced pendulum", *J. Stat. Phys.*, vol. 24, pp. 375–388, 1981.

[2] X. Xu, M. Wiercigroch, and M. P. Cartmell, "Rotating orbits of a parametrically-excited pendulum", *Chaos Soliton. Fract.*, vol. 23, pp. 1537–1548, 2005.

[3] X. Xu and M. Wiercigroch, "Approximate analytical solutions for oscillatory and rotational motion of a parametric pendulum", *Nonlinear Dyn.*, vol. 47, pp. 311–320, 2007.

[4] S. Lenci, E. Pavlovskaia, G. Rega, and M. Wiercigroch, "Rotating solutions and stability of parametric pendulum by perturbation method", *J. Sound Vib.*, vol. 310, pp. 243–259, 2008.

[5] M. J. Clifford and S. R. Bishop, "Rotating periodic orbits of the parametrically excited pendulum", *Phys. Lett. A*, vol. 201, pp. 191–196, 1995.

[6] B. Horton, J. Sieber, J. M. T. Thompson, M. Wiercigroch, "Dynamics of the nearly parametric pendulum", *Int. J. Nonlin. Mech.*, vol. 46, pp. 436–442, 2011.

[7] K. Nandakumar, M. Wiercigroch, and A. Chatterjee, "Optimum energy extraction from rotational motion in a parametrically excited pendulum", *Mech. Res. Commun.*, vol. 43, pp. 7–14, 2012.