Mathematical Derivation of Scattering Geometrical Optics from a Sphere by Modified Edge Representation Line Integral Around the Stationary Phase Point

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1. Introduction

Equivalent Edge Current (EEC)[1] realizes the reduction of PO surface radiation integral to line integral. The authors have been proposing the Modified Edge Representation (MER)[2] for deriving EECs. MER is unique in that it defines EECs not only along the periphery but also everywhere on the scatterer surface. At the Stationary Phase Point (SPP), the MER-EEC has the singularity. Authors have found numerically that the MER line integral around the SPP (MER-SPP) converges to Scattering Geometrical Optics (SGO) for variety of curved surfaces[3]. For a planer surface with one Stationary Phase Point (SPP), it has been proved mathematically[3]. For curved surfaces, SGO can be represented by MER-SPP only if the reflection wave front is spherical[4]. In this article, the authors discuss this “spherical reflection condition” in detail. Firstly, it is confirmed numerically that there is no frequency dependence on SGO extraction errors by MER-SPP[5]. Then the mathematical proof for SGO extraction from a curved surface by MER-SPP is studied. For simplicity, a sphere is considered as the scatterer in this article.

2. Modified Edge Representation Line Integral

The modified edge vector $\hat{\boldsymbol{\tau}}$ (Fig.1) is defined at every point on the boundary of the scatterer. It satisfies the diffraction law shown in Eq.1. The vector $\hat{\boldsymbol{t}}$ is the tangential vector to the real edge of the boundary of the scatterer. The vector $\hat{\boldsymbol{\tau}}$ generally differs from the real edge $\hat{\boldsymbol{t}}$. Only where the diffraction phenomenon occurs the vector $\hat{\boldsymbol{\tau}}$ and the vector $\hat{\boldsymbol{t}}$ have the same direction. The vector $\hat{n}$ is the unit normal vector on the surface of the scatterer and the vector $\hat{\boldsymbol{r}}_i$, $\hat{\boldsymbol{r}}_o$ is given by the direction of integration point to source and observer.

$$\begin{aligned}(\hat{\boldsymbol{r}}_i + \hat{\boldsymbol{r}}_o)\cdot\hat{\boldsymbol{\tau}} &= 0\\ \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\tau}} &= 0\end{aligned} \tag{1}$$

MER line currents are defined in Eq.2. MER scattering fields can be calculated by Eq.3, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are defined in Eq.4. The currents in the MER line integral are approximated $\boldsymbol{J} = 2\hat{\boldsymbol{n}}\times\boldsymbol{H_i}$, $\boldsymbol{M} = 0$ as the one of Physical Optics (PO), where $\boldsymbol{H_i}$ is the unperturbed incident magnetic field. PO current $\boldsymbol{J}$ includes only the radiation term and the relation between $\boldsymbol{J}, \boldsymbol{M}$ and $\boldsymbol{J_0}, \boldsymbol{M_0}$ is as follows: $\boldsymbol{J} = k\boldsymbol{J_0}$, $\boldsymbol{M} = k\boldsymbol{M_0}$.

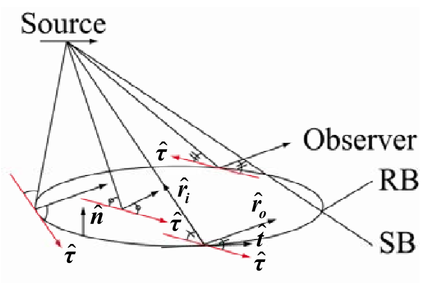

Fig.1 Modified Edge

$$\begin{aligned}\begin{bmatrix}\boldsymbol{J}^{\boldsymbol{I}}_{MER}\\ \boldsymbol{M}^{\boldsymbol{M}}_{MER}\end{bmatrix} &= \frac{\left\{\hat{\boldsymbol{r}}_o\times(\hat{\boldsymbol{r}}_o\times\begin{bmatrix}\boldsymbol{J}_0\\ \boldsymbol{M}_0\end{bmatrix})\right\}\cdot\hat{\boldsymbol{\tau}}}{j(1-(\hat{\boldsymbol{r}}_o\cdot\hat{\boldsymbol{\tau}})^2)(\hat{\boldsymbol{r}}_i+\hat{\boldsymbol{r}}_o)\cdot(\hat{\boldsymbol{\tau}}\times\hat{\boldsymbol{n}})}\hat{\boldsymbol{t}}\\ \begin{bmatrix}\boldsymbol{M}^{\boldsymbol{I}}_{MER}\\ \boldsymbol{J}^{\boldsymbol{M}}_{MER}\end{bmatrix} &= \frac{(\hat{\boldsymbol{r}}_o\times\begin{bmatrix}-\boldsymbol{J}_0\\ \boldsymbol{M}_0\end{bmatrix})\cdot\hat{\boldsymbol{\tau}}}{j(1-(\hat{\boldsymbol{r}}_o\cdot\hat{\boldsymbol{\tau}})^2)(\hat{\boldsymbol{r}}_i+\hat{\boldsymbol{r}}_o)\cdot(\hat{\boldsymbol{\tau}}\times\hat{\boldsymbol{n}})}\hat{\boldsymbol{t}}\end{aligned} \tag{2}$$

$$\boldsymbol{E}^{\boldsymbol{S}} = jk\eta\,\hat{\boldsymbol{r}}_o\times\hat{\boldsymbol{r}}_o\times(\boldsymbol{A}+\boldsymbol{A}') + jk\hat{\boldsymbol{r}}_o\times(\boldsymbol{B}+\boldsymbol{B}') \tag{3}$$

$$\begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{B} \\ \boldsymbol{A'} \\ \boldsymbol{B'} \end{bmatrix} = \frac{1}{4\pi}\oint \begin{bmatrix} \boldsymbol{J}^{I}_{MER} \\ \boldsymbol{M}^{I}_{MER} \\ \boldsymbol{J}^{M}_{MER} \\ \boldsymbol{M}^{M}_{MER} \end{bmatrix} \frac{e^{-jkr_o}}{r_o} \cdot d\Gamma \tag{4}$$

For the case where there is no SPP on the scatterer surface, it was mathematically proved that PO radiation integral is reduced to the MER line integral along the boundary of the scatterer using Stoke's theorem as well as the high frequency approximation.

$$\boldsymbol{E}^{S}_{PO} = \iint_{S} \boldsymbol{J}_{PO} \cdot \vec{\boldsymbol{G}}\, d\boldsymbol{S} = \int_{\Gamma}^{PER} \boldsymbol{J}^{PER}_{MER} \cdot \vec{\boldsymbol{G}}\, d\boldsymbol{l} = \boldsymbol{E}^{PER}_{MER} \tag{5}$$

For the case there exists SPP on the planar surface, PO radiation integral is reduced to the sum of two MER line integrals shown in Eq6. Where S' is an infinitesimal small region which contains SPP and S_0 is its complement. Γ is the integration path around the periphery of the scatterer and Γ' is the one around the SPP. For a curved surface, not only PO reduction to MER line integral collapses but also $\boldsymbol{E}^{SPP}_{MER}$ fails to represent SGO except the condition of spherical reflected wave front. In this article, only the latter is considered.

$$\boldsymbol{E}^{S}_{PO} = \boldsymbol{E}_{PO}\Big|_{S_0} + \boldsymbol{E}_{PO}\Big|_{S'} = \int_{\Gamma}^{PER} \boldsymbol{J}^{PER}_{MER} \cdot \vec{\boldsymbol{G}}\, d\boldsymbol{l} + \lim_{\rho \to 0} \int_{\Gamma'}^{SPP} \boldsymbol{J}^{SPP}_{MER} \cdot \vec{\boldsymbol{G}}\, d\boldsymbol{l} = \boldsymbol{E}^{PER}_{MER} + \boldsymbol{E}^{SPP}_{MER} \tag{6}$$

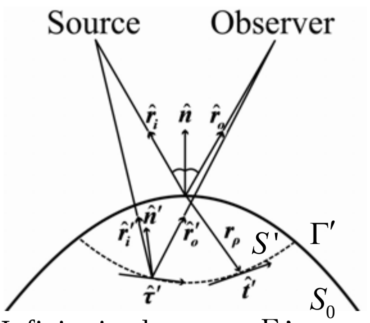

Fig.2 Infinitesimal contour Γ' around SPP

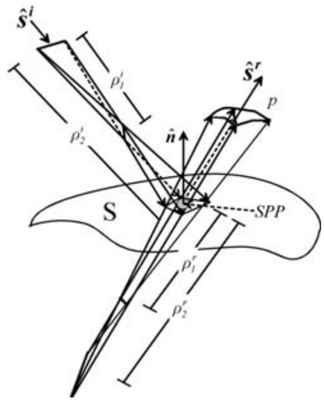

Fig.3 Astigmatic Ray Tubes
(Reflection From a Curved Surface)

3. Equivalence of MER-SPP and SGO

In this chapter, MER-SPP convergences to SGO for various surfaces are numerically investigated. Fig.4 shows the geometry of ellipsoid. As an example, parameters of ellipsoid are taken three cases as follows. R1=25 λ , R2=15 λ , 25 λ , 100 λ , R3=100 λ . X polarized dipole source is located at R=5 λ from SPP and observer is symmetry to Z axis ($\theta_i = \theta_r$) with its distance far enough from the scattering surface. Radius of integration path around SPP is ρ =0.00001 λ , which is infinitesimal enough to verify convergences. Results are shown in Fig4. Left side of vertical axis shows amplitudes of reflected rays normalized by the direct wave from dipole source including just only the radiation term. Right side of it shows the ratio of two radii of curvature of the reflected wave front. For the planar surface, SGO can be represented by MER-SPP at all observer angles. For the curved surfaces with its radius 10 λ , SGO can be represented by MER-SPP only under the condition of $\rho_1^r / \rho_2^r = 1$, which intends the spherical wave front. SGO extraction error by MER-SPP is directly related to the ratio of two radii of curvature of the reflected wave front.

4. Frequency Dependence of SGO Extraction Error by MER-SPP

In this Chapter, frequency dependence of SGO extraction error by MER-SPP is investigated. Two extreme cases are considered, which are $f = f_0,\ 0.001 f_0, 1000 f_0$. Parameters of ellipsoid at frequency f_0 are R1=25 λ , R2=100 λ and R3=100 λ , R=5 λ . Radius of integration path around SPP is ρ =0.00001 λ for all frequencies, which is small enough for the convergences. Results are shown in Fig6. MER-SPP, SGO and the ratio of two radii of curvature of the reflected wave front is also plotted. Three results for the different frequency are indistinguishable and irrelevant to frequencies. SGO extraction condition by MER-SPP seems to be only the geometrical one that reflection wave front is spherical and therefore frequency independent. This suggests that the exponential term of MER-SPP (Eq.3) can be taken out from integrand and mathematical treatment will become simple for identifying the equivalence analytically.

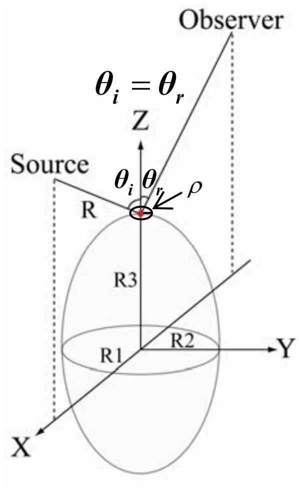

Fig. 4 Geometry of Ellipsoid

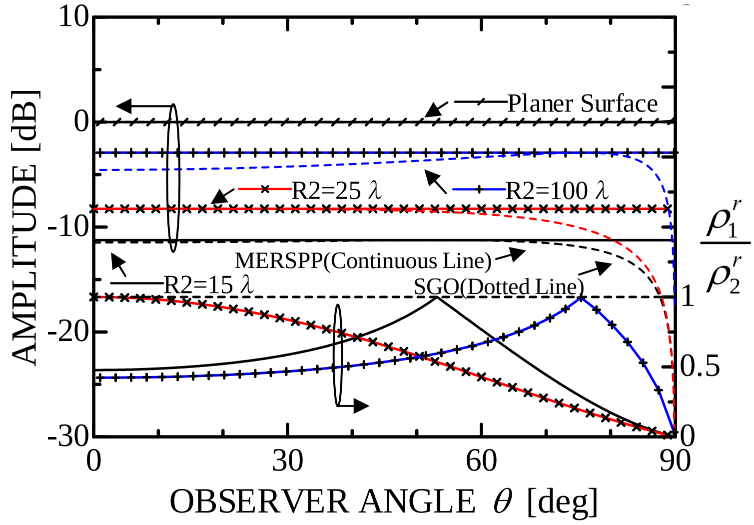

Fig. 5 MER-SPP Convergence to SGO for Various Surfaces

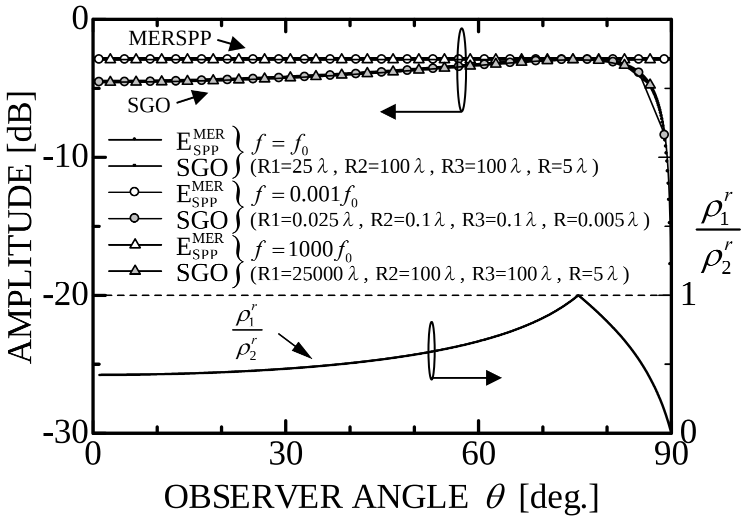

Fig. 6 Frequency Independence of SGO Extraction Error by MER-SPP

5. Mathematical Discussion of SGO by MER-SPP

In this chapter, the extraction of SGO by MER-SPP for the reflection with the spherical wave front is mathematically discussed based on the results in Chapter4.

First of all, when the wave front is spherical, SGO reflected field can be written as:

$$\begin{aligned} \boldsymbol{E}_{SGO} &= \boldsymbol{E}^{i}_{GO}(SPP)\cdot R\sqrt{\frac{\rho_1^r\rho_2^r}{(\rho_1^r+r)(\rho_2^r+r)}}\,e^{-jkr} \\ &= \boldsymbol{E}^{i}_{GO}(SPP)\cdot R\,\frac{a}{2(r+a)}\,e^{-jkr}\left(\because \rho_1^r=\rho_2^r=\frac{ar}{a+2r}\right) \end{aligned} \tag{7}$$

where $\rho_1^r, \rho_2^r =$ principal radii of curvature of the reflected wave front at the point of reflection

R = reflection coefficient

Secondly, MER-SPP is expressed as the line integral around SPP taking the limit of $\rho \rightarrow 0$.

$$\boldsymbol{E}^{SPP}_{MER} = \lim_{\rho\to 0} j\frac{k\eta}{4\pi}\oint_{\Gamma'}\hat{\boldsymbol{r}}'_o\times\left[\hat{\boldsymbol{r}}'_o\times\boldsymbol{J}_{MER}+\boldsymbol{M}_{MER}\right]\frac{e^{-jk(r'_i+r'_o)}}{r'_o}\cdot d\boldsymbol{\Gamma}' \tag{8}$$

According to Chapter4, exponential term can be taken out from integrand. At this stage, the reflected wavefront is general and can be astigmatic. The vector on the integration path around SPP is expressed by the one at the SPP. Eq.9 is derived using $\hat{\boldsymbol{r}}_o' \times \left[\hat{\boldsymbol{r}}_o' \times \boldsymbol{J}_{MER} \right] = \boldsymbol{F} \dfrac{1}{(\hat{\boldsymbol{r}}_i' + \hat{\boldsymbol{r}}_o') \cdot (\hat{\boldsymbol{\tau}}' \times \hat{\boldsymbol{n}}')}$, $\hat{\boldsymbol{r}}_o' \times \boldsymbol{M}_{MER} = \boldsymbol{G} \dfrac{1}{(\hat{\boldsymbol{r}}_i' + \hat{\boldsymbol{r}}_o') \cdot (\hat{\boldsymbol{\tau}}' \times \hat{\boldsymbol{n}}')}$ and it works out regardless of the scatterer.

$$
\begin{aligned}
\boldsymbol{E}_{MER}^{SPP} &= \lim_{\rho \to 0} j \frac{k\eta}{4\pi} \frac{e^{-jk(r_i'+r_o')}}{r_o'} \oint_{\Gamma'} \frac{\boldsymbol{F}+\boldsymbol{G}}{(\hat{\boldsymbol{r}}_i' + \hat{\boldsymbol{r}}_o') \cdot (\hat{\boldsymbol{\tau}}' \times \hat{\boldsymbol{n}}')} \cdot \boldsymbol{d\Gamma'} \\
&= \lim_{\rho \to 0} j \frac{k\eta}{4\pi} \frac{e^{-jk(r_i'+r_o')}}{r_o'} \oint_{\Gamma'} \frac{\boldsymbol{F}+\boldsymbol{G}}{\left| (\hat{\boldsymbol{r}}_i' + \hat{\boldsymbol{r}}_o') \times \hat{\boldsymbol{n}}' \right|} \cdot \boldsymbol{d\Gamma'} \qquad \left(\because \hat{\boldsymbol{\tau}}' = \frac{(\hat{\boldsymbol{r}}_i' + \hat{\boldsymbol{r}}_o') \times \hat{\boldsymbol{n}}'}{\left| (\hat{\boldsymbol{r}}_i' + \hat{\boldsymbol{r}}_o') \times \hat{\boldsymbol{n}}' \right|} \right) \\
&= \lim_{\rho \to 0} j \frac{k\eta}{4\pi} \frac{e^{-jk(r_i'+r_o')}}{r_o'} \oint_{\Gamma'} \frac{\boldsymbol{F}+\boldsymbol{G}}{\left| \left(\hat{\boldsymbol{r}}_i + \hat{\boldsymbol{r}}_o + \left[\left(\dfrac{\hat{\boldsymbol{r}}_i \cdot \hat{\boldsymbol{r}}_\rho}{r_i} \right) \hat{\boldsymbol{r}}_i + \left(\dfrac{\hat{\boldsymbol{r}}_o \cdot \hat{\boldsymbol{r}}_\rho}{r_o} \right) \hat{\boldsymbol{r}}_o - \left(\dfrac{1}{r_i} + \dfrac{1}{r_o} \right) \hat{\boldsymbol{r}}_\rho \right] \rho \right) \times \hat{\boldsymbol{n}}' \right|} \cdot \boldsymbol{d\Gamma'}
\end{aligned}
\tag{9}
$$

where $\hat{\boldsymbol{n}}'$ is the unit normal vector on the integration path around SPP and $\boldsymbol{F}+\boldsymbol{G} = j\,\hat{\boldsymbol{r}}_o' \times \hat{\boldsymbol{r}}_o' \times \boldsymbol{J}_o = -\dfrac{2(\hat{\boldsymbol{r}}_o' \cdot \hat{\boldsymbol{n}}')}{4\pi r_i'} \hat{\boldsymbol{r}}_o' \times \left(\boldsymbol{p}' \times \hat{\boldsymbol{r}}_o' \right)$

For simplicity, let us consider the case of $r_i = r_o = r$ and the spherical surface with the radius a. The unit normal vector $\hat{\boldsymbol{n}}'$ is written as: $\hat{\boldsymbol{n}}' = \left(\rho\cos(\xi)/a,\ \rho\sin(\xi)/a,\ \sqrt{1-(\rho/a)^2} \right) \quad (0 \le \xi \le 2\pi)$

$$
\boldsymbol{E}_{MER}^{SPP} = \lim_{\rho \to 0} \frac{-jk\eta}{4\pi} \frac{e^{-jk(r_i'+r_o')}}{r_o'} \frac{1}{4\pi r_i'} \hat{\boldsymbol{r}}_o' \times \left(\boldsymbol{p}' \times \hat{\boldsymbol{r}}_o' \right) \int_\xi \frac{ar(\cos\theta + \rho\sin\theta\sin\xi/a)}{\sqrt{\sin^2(\xi)\cos^2\theta\left(r + a\cos\theta\right)^2 + \cos^2(\xi)\left(r\cos\theta + a\right)^2}} \cdot \boldsymbol{d\xi} \tag{10}
$$

The condition of spherical wave front for its surface sphere is $\theta = \mathrm{Arccos}\sqrt{a_2/a_1} = \mathrm{Arccos}\,1$ =0deg for $a_1, a_2 =$ a. Now, $\boldsymbol{E}_{MER}^{SPP}$ is written as Eq11 and it is proved that SGO can be represented by MER-SPP.

$$
\begin{aligned}
\boldsymbol{E}_{MER}^{SPP} &= -j \frac{k\eta}{4\pi} \frac{e^{-jkr}}{r} \frac{1}{4\pi} \hat{\boldsymbol{r}}_o \times \left(\boldsymbol{p}' \times \hat{\boldsymbol{r}}_o \right) \frac{a}{(r+a)} e^{-jkr} \int_\xi d\xi \\
&= -j \frac{k\eta}{4\pi} \frac{e^{-jkr}}{r} \hat{\boldsymbol{r}}_o \times \left(\boldsymbol{p}' \times \hat{\boldsymbol{r}}_o \right) \frac{a}{2(r+a)} e^{-jkr} \\
&= \boldsymbol{E}_{GO}^{i} \cdot R \, \frac{a}{2(r+a)} e^{-jkr} \\
&= \boldsymbol{E}_{GO}^{r}
\end{aligned}
\tag{11}
$$

6. Conclusion

SGO from a curved surface by MER-SPP is mathematically discussed under the condition of spherical wave reflection. For simplicity, a sphere is considered as the scatterer and it is proved that SGO can be represented by MER-SPP. Similar mathematical discussion for another scatterer will be considered. Further investigation on SGO extraction errors by MER-SPP in general case is still left for further study.

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