Modern Mathematics in Antenna Arrays

Hiroaki MIYASHITA and Yoshihiko KONISHI

Mitsubishi Electric Corporation
5-1-1 Ofuna, Kamakura, Kanagawa 247-8501 Japan
E-mail: miyas@ieee.org

1 Introduction

Recent remarkable developments in mathematical physics are encouraged by borderless interactions between pure mathematics and physics. For example, even in 'pure' classical physics such as mechanics is much influenced by analyses on manifolds including differential geometry and topology[1]. There, one of the central concepts is the Lagrangian submanifolds, and it even found a way in asymptotic analyses of the wave phenomena such as the Maslov's method[2]. Because the Maxwell's equations are partial differential equations, it is quite natural that the initial approaches were based on calculus. However, to the authors' impression, applications of algebraic techniques are very limited for antenna theories while a number of available mathematical tools do exist. In this talk, some trials are described. It is shown that there are rich algebraic structures in the antenna arrays. Although mathematical techniques used here are well established and even 'classic' for mathematicians, it is hoped that antenna engineers may find something 'modern' flavor.

2 Absolute Arrays of Arrays Principle

Let us summarize the previously reported result called the absolute arrays of arrays principle[3]. It provides general frameworks to attack the antenna arrays as mathematical objects.

An array factor, *i.e.* Schelkunoff polynomial $F_P(z)$, of a linear antenna array with P radiating elements is expressed as follows[4]:

$$F_P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{P-1} z^{P-1},$$

where $a_m,\ (m = 0, 1, 2, \cdots, P-1)$ are complex array excitation coefficients. We assume that the position of m-th element is md, the observation angle is θ from the boresight, and the array is operated at the wavelength λ. Then z is given as follows:

$$\begin{aligned} z &= e^{2\pi j u}, \\ u &= \frac{d}{\lambda}\sin\theta, \end{aligned}$$

where u is the universal parameter. In case that a beam is steered to an angle θ_0, the corresponding phase term can be conveniently extracted from a_m as,

$$u = \frac{d}{\lambda}(\sin\theta - \sin\theta_0).$$

If $F_P(z)$ is factorized into,

$$F_P(z) = F_{p_1}^{(1)}(z) F_{p_2}^{(2)}(z^{p_1}) \cdots F_{p_n}^{(n)}(z^{p_1 p_2 \cdots p_{n-1}}),$$

the arrays of arrays principle is readily applied by considering the subarrays as radiating elements, where the m-th subarray factor with K elements is given by the following form,

$$F_K^{(m)}(z) = a_0^{(m)} + a_1^{(m)} z + \cdots + a_{K-1}^{(m)} z^{K-1},$$

with $a_i^{(m)} \in \mathbb{C},\ (0 \leq i \leq K-1)$.

To implement 'modern' mathematics, we need to observe that (1) is a product of functions over covering spaces of algebraic curves[5][6]:

$$z \xrightarrow{h_{p_1}} z^{p_1} \xrightarrow{h_{p_2}} z^{p_1 p_2} \xrightarrow{h_{p_3}} \cdots \xrightarrow{h_{p_{n-1}}} z^{p_1 p_2 \cdots p_{n-1}}, \tag{1}$$

where $h_n : \mathbb{C}^* \to \mathbb{C}^*;\ z \mapsto z^n$, and $\mathbb{C}^*$ is defined to be the set $\mathbb{C} - \{0\}$. Each h_n gives unbranched covering[6] and the tower results from Kummer extensions[7] of algebraic functions. It is a tower of Galois coverings[5][6] and its structure is completely determined by the Jordan=Hölder theorem[7]. The following theorem is obtained.

[General Array Factorization Theorem][3] A Schelkunoff polynomial of a linear array with P elements can be regarded as a function over a tower of Galois coverings. Arrays of Arrays Principle is determined by a projective system of $\mathbb{Z}/P\mathbb{Z}$. In particular, the system which corresponds a composition series of $\mathbb{Z}/P\mathbb{Z}$ gives the longest tower of the Galois coverings.

To treat the case of infinite arrays, algebraic geometry is applied. The corresponding Galois coverings X are governed by the Grothendieck's étale fundamental group $\pi_1^{\mathrm{et}}(X)$[8] which is regarded as the absolute Galois group[9]. In this case, $\pi_1^{\mathrm{et}}(X) = \hat{\mathbb{Z}}$, where $\hat{\mathbb{Z}}$ is the profinite completion of the set of integers regarded as an additive group. Accounting freedom of all possible implementation of feeding networks and partial expansion of factorized subarray pattern (sub-product[10]), and attributing the operation to the symbol $\mathcal{A}_{\mathcal{P}}$, we finally have the following 'Absolute Arrays of Arrays Principle,'

$$\mathcal{A}_{\mathcal{P}}\,\hat{\mathbb{Z}}.$$

The quantity generates all possible topologies of array pattern factorization, including the topologies of the feeding networks, in the category of towers of Galois coverings.

3 Duality in Antenna Arrays

The approach in the previous section is very much geometric: Application of the theory of covering spaces. However, purely algebraic approach is also possible because the structure of a uniformly spaced linear array is rather simple. The alternative tool is the duality of Abelian groups. First of all, let us summarize. Let G and $\mathbb{C}^\times$ be an Abelian group and a multiplicative group of complex numbers, respectively. Homomorphisms $\tilde{G} = \mathrm{Hom}(G, \mathbb{C}^\times)$ from G to $\mathbb{C}^\times$ constitute Abelian groups called *characters* of G or the dual of G. If written additively, the expression $\tilde{G} = \mathrm{Hom}(G, \mathbb{R}/\mathbb{Z})$ is homeomorphic to the former. The following relations exit in the additive notation,

[Duality of Abelian Groups][11] Let G be a finite additive Abelian group. If subgroups $H \subset G$ and $\Phi \subset \tilde{G}$ satisfy the following relations,

$$\begin{aligned} H^\perp &= \{\chi \in \tilde{G} \mid \chi(h) = 0 \quad (\forall h \in H)\}, \\ \Phi^\perp &= \{x \in G \mid \chi(x) = 0 \quad (\forall \chi \in \Phi)\}, \end{aligned}$$

then $H^\perp$ and $\Phi^\perp$ are subgroups of $\tilde{G}$ and G, respectively, and there are the following canonical homeomorphisms,

$$H^\perp \cong \widetilde{G/H}, \qquad \tilde{G}/H^\perp \cong \tilde{H}.$$

Furthermore, there are 'Galois' correspondences $H \longleftrightarrow H^\perp$, $\Phi^\perp \longleftrightarrow \Phi$ such that

$$(H^\perp)^\perp \cong H, \qquad (\Phi^\perp)^\perp \cong \Phi.$$

The set $G^\perp$ is called an *annihilator* of G. The application of the theorem to (1) is immediate. It is known that n in the map $h_n : \mathbb{C}^* \to \mathbb{C}^*;\ z \mapsto z^n$ contributes to array arrangement[3]. The tower (1) gives the following sequence of additive groups,

$$\mathbb{Z} \supset p_1\mathbb{Z} \supset p_1p_2\mathbb{Z} \supset p_1p_2p_3\mathbb{Z} \supset \cdots$$

The corresponding annihilators in the character groups are as follows:

$$\mathbb{R}/\mathbb{Z} \subset \mathbb{R}/p_1\mathbb{Z} \subset \mathbb{R}/p_1p_2\mathbb{Z} \subset \mathbb{R}/p_1p_2p_3\mathbb{Z} \subset \cdots$$

If $G \supset H$ and the corresponding annihilators are X/G and X/H, respectively, where X is the annihilator of the unit element of G, then the 'Galois' group of $X/G \subset X/H$ is given by G/H[5][6]. In the above case, for example, the Galois group of the first part $\mathbb{R}/\mathbb{Z} \subset \mathbb{R}/p_1\mathbb{Z}$ is $\mathbb{Z}/p_1\mathbb{Z}$. With this observation, the previously reported theory [3] can be recovered with simpler considerations. For other aspects including the physical meaning of the action of Galois groups will not be repeated here.

4 Pontryagin Duality, Schelkunoff Circle, and Torus

In this section, we try to understand mathematical counterpart of the Schelkunoff's unit circle[4] $\mathbb{T} = \{z \in \mathbb{C}^\times \mid |z| = 1\} \simeq \mathbb{R}/\mathbb{Z}$. $\mathbb{T}$ is also called one dimensional torus. A n dimensional torus $\mathbb{T}^n$ can be also defined by taking its direct product n times. Theory of topological groups gives the relation[11],

$$\tilde{\mathbb{Z}} \cong \mathbb{T}.$$

The following theorem is well-known,

[Pontryagin Duality Theorem][11] Let G be a locally compact Abelian group. A map $\eta : G \to \tilde{G}$ defined by

$$\eta(x)(\chi) = \chi(x)$$

for $x \in G$ and $\chi \in \tilde{G}$ gives canonical algebraic and topological homeomorphisms.

Therefore $\tilde{\tilde{Z}} \cong \tilde{\mathbb{T}} \cong \mathbb{Z}$, for example. The $\tilde{\mathbb{Z}} \cong \mathbb{T}$ says that the dual of a discrete group $\mathbb{Z}$ is a compact group $\mathbb{T}$, and $\tilde{\mathbb{T}} \cong \mathbb{Z}$ has the opposite property. For a uniformly spaced linear array, $m \in \mathbb{Z}$ can be regarded as the m-th a position of radiating elements. Therefore the discrete group $\mathbb{Z}$ represents the array arrangement and the Schelkunoff's unit circle $\mathbb{T}$ is its compact dual. More explicitly, the array arrangement is $\mathbb{Z}$ in the universal parameter space u, and the character is homeomorphic to $\mathbb{T}$, the Schelkunoff's unit circle, because

$$\mathrm{Hom}(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z},$$

by choosing the map from the generating element $1 \in \mathbb{Z}$ to a point on $\mathbb{T}$. It is also true for $\mathbb{Z}^n$, n times direct product of $\mathbb{Z}$, there is a homeomorphism,

$$\widetilde{\mathbb{Z}^n} \cong \mathbb{T}^n.$$

In particular, we can define a 'Schelkunoff torus' $\mathbb{T}^2 \cong \mathbb{T} \times \mathbb{T}$ for a two dimensional array. It is exactly a torus in our ordinary sense.

5 Conclusion

Trials are reported for attacking the antenna array theory by algebraic techniques. It is shown that the duality theorems of Abelian groups provides a simple view of the Schelkunoff's unit circle as a compact dual group of a discrete array arrangement. The authors hope that engineers will pay more attention to algebra and extensively apply the methodology in antenna theory.

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