Analysis of Longitudinally Inhomogeneous Waveguides By Cascading Short Linear Sections

Mohammad Khalaj-Amirhosseini
College of Electrical Engineering - Iran University of Science and Technology – Tehran – Iran
Email: khalaja@iust.ac.ir

Abstract

A new method is introduced to frequency domain analyze arbitrary Longitudinally Inhomogeneous Waveguides (LIWs), in this paper. The LIWs are subdivided into several short linear sections instead of uniform ones. The chain parameter matrix of linear sections is obtained by expressing the electric and magnetic fields in power series expansion. This method is applicable to all arbitrary LIWs. The accuracy of the proposed method is verified using a comprehensive example.

Keywords: Longitudinally Inhomogeneous Waveguides, Linear Sections, Power Series Expansion.

1. Introduction

Longitudinally Inhomogeneous Waveguides (LIWs) can be used in microwaves as phase changers, matching transformers and filters [1-3], especially for high power applications. The differential equations describing LIWs have non-constant coefficients and so except for a few special cases no analytical solution exists for them. There are some methods to analyze the LIWs such as finite difference [4], Taylor's series expansion [5], Fourier series expansion [6], the method of Moments [7] and the equivalent sources [8]. Of course, the conventional and most straightforward method is subdividing LIWs into many short uniform sections [9]. In this paper, the conventional method is modified by subdividing LIWs into many short linear sections instead of uniform ones. In the proposed method, the permittivity function of LIWs is assumed to vary linearly between two ends of the short sections. Also, the distribution of the electric and magnetic fields along the LIWs are expanded in power series and their unknown coefficients are related to each other by some recursive relations. This method is applicable to all arbitrary LIWs. The accuracy of the proposed method is studied using a comprehensive example.

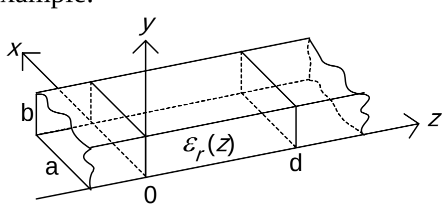

Figure 1: A typical LIW

2. The Equations of LIWs

Fig. 1 shows a typical LIW with dimensions *a* and *b*, filled by an inhomogeneous lossy dielectric with complex electric permittivity $\varepsilon_r(z)$ and length *d*. It is assumed that a TE_{10} mode with electric filed strength E^i propagates towards the positive *z* direction. The differential equations describing LIWs are given by

$$\frac{dE_y(z)}{dz} = j\omega\mu_0 H_x(z) \tag{1}$$

$$\frac{dH_x(z)}{dz} = j\omega\varepsilon_0\left(\varepsilon_r(z) - (f_c/f)^2\right)E_y(z) \tag{2}$$

where f_c is the cutoff frequency of the hollow waveguide. Furthermore, the terminal conditions for LIWs are as follows

$$E_y(0) - Z_S H_x(0) = 2E^i \tag{3}$$

$$E_y(d) + Z_L H_x(d) = 0 \tag{4}$$

where

$$Z_S = Z_L = Z_{TE} = \begin{cases} \dfrac{\eta_0}{\sqrt{1-(f_c/f)^2}} \ , & f > f_c \\ j\dfrac{\eta_0}{\sqrt{(f_c/f)^2-1}}, & f < f_c \end{cases} \tag{5}$$

is the waveguide impedance, in which $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$ is the wave impedance of the free space.

3. Linear Approximation

The analysis of LIWs using linear approximation is introduced, in this section. It is assumed that the electric permittivity of LIWs could be expressed by a linear approximation as follows

$$\varepsilon_r(z) \cong \varepsilon_r(0) + (\varepsilon_r(d) - \varepsilon_r(0))(z/d) = P_0 + P_1(z/d) \tag{6}$$

Also, we can consider the electric and magnetic fields in power series as follows

$$E_y(z) = \sum_{n=0}^{\infty} E_n (z/d)^n \tag{7}$$

$$H_x(z) = \sum_{n=0}^{\infty} H_n (z/d)^n \tag{8}$$

in which the coefficients E_n and H_n are unknown coefficients. Using (6)-(8) in (1)-(2), the following relations are obtained.

$$\sum_{n=0}^{\infty} (n+1) E_{n+1} (z/d)^n = d \sum_{k=0}^{\infty} Z_0 H_k (z/d)^k \tag{9}$$

$$\sum_{n=0}^{\infty} (n+1) H_{n+1} (z/d)^n = d \sum_{k=0}^{\infty} \left(Y_0 E_k (z/d)^k + Y_1 E_k (z/d)^{k+1} \right) \tag{10}$$

where

$$Z_0 = j\omega\mu_0 \tag{11}$$

$$Y_0 = j\omega\varepsilon_0 \left(P_0 - (f_c/f)^2 \right) \tag{12}$$

$$Y_1 = j\omega\varepsilon_0 P_1 \tag{13}$$

Equating the coefficients of the same power terms in two sides of (9)-(10), gives us the following recursive relations for $n = 0, 1, 2, \ldots$

$$E_{n+1} = \frac{d}{n+1} Z_0 H_n \tag{14}$$

$$H_{n+1} = \frac{d}{n+1} \left(Y_0 E_n + Y_1 E_{n-1} \right) \tag{15}$$

Of course, all unknown coefficients can be related to two coefficients E_0 and H_0. For example, some of the coefficients are obtained as follows

$$E_1 = d Z_0 H_0 \tag{16}$$

$$H_1 = d Y_0 E_0 \tag{17}$$

$$E_2 = \frac{d^2}{2} Z_0 Y_0 E_0 \tag{18}$$

$$H_2 = \frac{d^2}{2} Y_0 Z_0 H_0 + \frac{d}{2} Y_1 E_0 \tag{19}$$

$$E_3 = \frac{d^2}{6} Z_0 Y_1 E_0 + \frac{d^3}{6} Z_0^2 Y_0 H_0 \tag{20}$$

$$H_3 = \frac{d^2}{3} Y_1 Z_0 H_0 + \frac{d^3}{6} Y_0^2 Z_0 E_0 \tag{21}$$

$$E_4 = \frac{d^4}{24} (Z_0 Y_0)^2 E_0 + \frac{d^3}{12} Z_0^2 Y_1 H_0 \tag{22}$$

$$H_4 = \frac{d^4}{24} (Y_0 Z_0)^2 H_0 + \frac{d^3}{6} Y_1 Z_0 Y_0 E_0 \tag{23}$$

Finally, we can find the chain parameter matrix of LIWs as follows

$$\begin{bmatrix} E_y(d) \\ -H_x(d) \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{\infty} E_n \\ -\sum_{n=0}^{\infty} H_n \end{bmatrix} = \mathbf{\Phi} \begin{bmatrix} E_y(0) \\ -H_x(0) \end{bmatrix} = \mathbf{\Phi} \begin{bmatrix} E_0 \\ -H_0 \end{bmatrix} \tag{24}$$

Substituting the obtained coefficients E_n and H_n in (24) and after some mathematical manipulations, one can obtain the chain parameter matrix of LIWs as follows

$$\mathbf{\Phi} = \mathbf{\Phi}_{\text{unif}} + \Delta\mathbf{\Phi} \tag{25}$$

where

$$\mathbf{\Phi}_{\text{unif}} = \begin{bmatrix} 1 + \sum_{n=1}^{\infty} \frac{d^{2n}}{(2n)!}(Z_0 Y_0)^n & -\left(d + \sum_{n=1}^{\infty} \frac{d^{2n+1}}{(2n+1)!}(Z_0 Y_0)^n \right) Z_0 \\ -\left(d + \sum_{n=1}^{\infty} \frac{d^{2n+1}}{(2n+1)!}(Y_0 Z_0)^n \right) Y_0 & 1 + \sum_{n=1}^{\infty} \frac{d^{2n}}{(2n)!}(Y_0 Z_0)^n \end{bmatrix} = \exp\left(-\begin{bmatrix} 0 & Z_0 d \\ Y_0 d & 0 \end{bmatrix} \right) \tag{26}$$

is the chain parameter matrix of uniform waveguides assuming $\varepsilon_r(z) = P_0$ [9]. Also, the matrix $\Delta\mathbf{\Phi}$ in (26) can be written as follows, ignoring the terms with power greater than four for d.

$$\Delta\mathbf{\Phi} \cong \begin{bmatrix} \frac{1}{6} d^2 Z_0 Y_1 + d^4 \left(\frac{1}{30} Z_0^2 Y_1 Y_0 + \frac{1}{180} Z_0^2 Y_1^2\right) & -\frac{1}{12} d^3 Z_0^2 Y_1 \\ -\frac{1}{2} d Y_1 - d^3 \left(\frac{1}{6} Y_1 Z_0 Y_0 + \frac{1}{30} Y_1^2 Z_0\right) & \frac{1}{3} d^2 Y_1 Z_0 + d^4 \left(\frac{1}{20} Z_0^2 Y_1 Y_0 + \frac{1}{72} Z_0^2 Y_1^2\right) \end{bmatrix} \tag{27}$$

We expect that the added matrix $\Delta\mathbf{\Phi}$ in (25) can modify the conventional method of analysis of LIWs. To analyze LIWs, we can subdivide them into K linear sections whose chain parameter matrix can be expressed using (25)-(27) but substituting $\Delta z = d/K$ instead of d. The chain parameter matrix corresponding to the terminals of LIWs can be written as the multiplication of the chain parameter matrices of all sections. Then, one can obtain the electric and magnetic fields at two ends of each section using the chain parameter matrix of terminals and the boundary conditions (3)-(4). It is worth to mention that we can also determine the electric and magnetic fields at any point located between two ends of each section using the relations (7)-(8) and (14)-(15) after knowing the electric and magnetic fields at two ends of all sections. Moreover, the elements of the chain parameter matrix can be used to find the S parameters as follows

$$S_{11} = \frac{-\mathbf{\Phi}(1,1) Z_{TE} - \mathbf{\Phi}(1,2) + \mathbf{\Phi}(2,1) Z_{TE}^2 + \mathbf{\Phi}(2,2) Z_{TE}}{\mathbf{\Phi}(1,1) Z_{TE} - \mathbf{\Phi}(1,2) - \mathbf{\Phi}(2,1) Z_{TE}^2 + \mathbf{\Phi}(2,2) Z_{TE}} \tag{28}$$

$$S_{21} = S_{12} = \frac{2 Z_{TE}}{\mathbf{\Phi}(1,1) Z_{TE} - \mathbf{\Phi}(1,2) - \mathbf{\Phi}(2,1) Z_{TE}^2 + \mathbf{\Phi}(2,2) Z_{TE}} \tag{29}$$

$$S_{22} = \frac{\mathbf{\Phi}(1,1) Z_{TE} - \mathbf{\Phi}(1,2) + \mathbf{\Phi}(2,1) Z_{TE}^2 - \mathbf{\Phi}(2,2) Z_{TE}}{\mathbf{\Phi}(1,1) Z_{TE} - \mathbf{\Phi}(1,2) - \mathbf{\Phi}(2,1) Z_{TE}^2 + \mathbf{\Phi}(2,2) Z_{TE}} \tag{30}$$

4. Verifying the Modification

In this section, we verify the modifications obtained from using linear sections instead of uniform ones in the analysis of LIWs. Consider a WRG-90 waveguide (a = 0.9 in and b = 0.4 in) filled by an exponential dielectric with the following electric permittivity function

$$\varepsilon_r(z) = \varepsilon_{r0} \exp(kz/d) \tag{31}$$

Now, assume that $\varepsilon_{r0} = 1-j0$, d = 2 cm and k = 1. A TE_{10} mode wave with frequency f = 10 GHz and the electric field strength E^i = 1.0 V/m propagates in the assumed LIW. Figs. 2-3, compare the amplitude and phase of the electric field obtained from the exact solution [7] and from the introduced and the conventional methods with K = 10 linear and uniform sections. One sees an excellent agreement between the results obtained from the introduced method with the exact ones. Also, Figs. 4-5 show the relative error corresponding to the scattering parameters S_{11} and S_{21} for the methods of cascading K uniform and linear sections versus K. It is seen that the accuracy of cascading linear sections is more than that of cascading uniform sections. Furthermore, if one considers more terms in (27) the accuracy of the cascading linear sections will be increased, certainly. According to this example, one may be satisfied about the modification of the introduced method. Also, it is obvious that the introduce method is applicable to arbitrary LIWs.

5. Conclusion

A new method was introduced to frequency domain analyze arbitrary Longitudinally Inhomogeneous Waveguides (LIWs). The LIWs are subdivided into several short linear sections instead of uniform ones. The chain parameter matrix of linear sections is obtained by expressing the electric and magnetic fields in power series expansion. The validity of the proposed method was verified using a comprehensive example. It was seen that the accuracy of cascading linear sections is more than that of cascading uniform sections. The introduced method can be extended for LIWs, whose magnetic permeability is inhomogeneous solely or along with their electric permittivity.

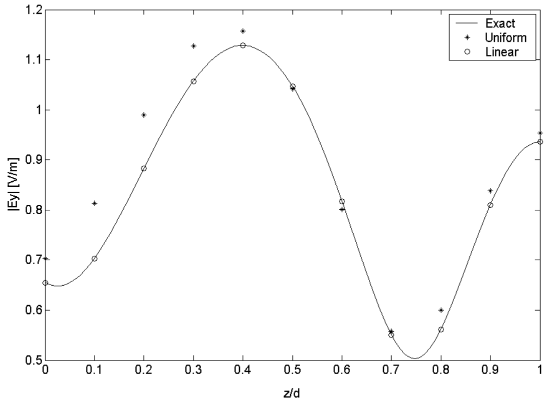

Figure 2. The amplitude of the electric field distribution

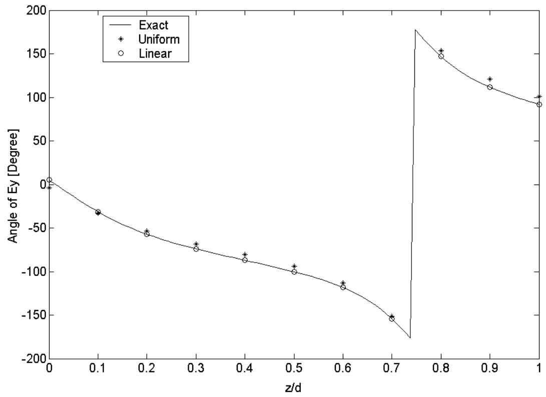

Figure 3. The phase of the electric field distribution

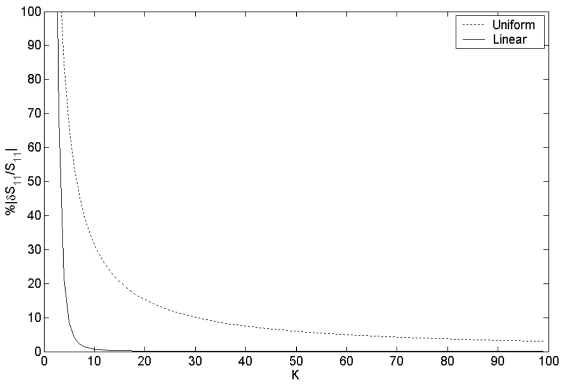

Figure 4. The relative error of the obtained S_{11} parameter

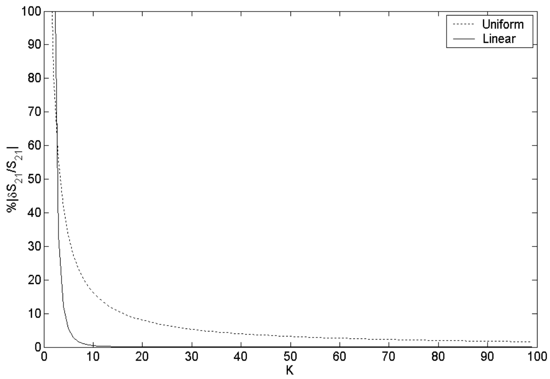

Fig. 5. The relative error of the obtained S_{21} parameter

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