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Controlling Method to Avoid Bifurcations of Periodic Points Using Maximum Lyapunov Exponent

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Abstract—This paper provides a novel controlling method to avoid improper bifurcations of stable fixed and periodic points using the maximum Lyapunov exponent. The Lyapunov exponents that can be calculated from the sequence of the points characterize the topological properties of a stable fixed or periodic point if it is the limit of the sequence. Our main ideas are observing the maximum Lyapunov exponent to predict a bifurcation caused by the change of any parameter value, and controlling any adjustable system-parameter value so that the bifurcation never appears. The proposed method can be led from an optimization problem on the maximum Lyapunov exponent. We presented not only its mathematical derivation but also the results of numerical experiments on avoiding bifurcations to demonstrate the validity of the proposed method.

1. Introduction

Recently, there has been increasing interest on controlling bifurcations of limit sets observed in dynamical systems. For example, bifurcation control [1] deals with modification of bifurcation characteristics of a nonlinear system by a designed control input. The objectives of typical bifurcation control include delaying the onset of an inherent bifurcation, changing the parameter value of an existing bifurcation point, and so on.

In contrast, we deal with the problem avoiding the occurrence of bifurcations in discrete-time dynamical systems. Here, we assume that an improper bifurcation of a stable fixed or periodic point can occur by forcible change of system parameter value and we cannot directly operate state variables. Since a method of state feedback control cannot be utilized, to avoid bifurcations we need another method changing adjustable system-parameter value.

The maximum Lyapunov exponent that can be calculated from the sequence of the points [2] characterize the stability of a fixed or periodic point if that is the limit of the sequence. Therefore, by computing the maximum Lyapunov exponent from a sequence, we can predict the occurrence of a bifurcation caused by forcible change of system parameter value. Moreover, the occurrence of a bifurcation can be avoided by controlling the maximum Lyapunov exponent. From the aforementioned assumption, feedback control of Lyapunov exponents [3] is unavailable.

In this paper, we provide a novel method to avoid improper bifurcations of stable fixed and periodic points using the maximum Lyapunov exponent. Our main ideas are predicting the occurrence of bifurcations by observing the maximum Lyapunov exponent, and controlling parameter values so that the bifurcations never appear. We describe our controlling method derived from a minimization problem with respect to the maximum Lyapunov exponent and show experimental results to demonstrate the validity of the proposed method.

2. Preparation

Let us consider the map $\boldsymbol{f}$ or the discrete-time dynamical system described by

$$\begin{aligned} &\boldsymbol{f} : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N \\ &\qquad (\boldsymbol{x}(t), \boldsymbol{p}(t)) \mapsto \boldsymbol{x}(t+1) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{p}(t)) \end{aligned} \tag{1}$$

where t is the discrete time, $\boldsymbol{x}(t) \in \mathbb{R}^N$ is the vector of state variables, $\boldsymbol{p}(t) \in \mathbb{R}^M$ is the vector of time-variant system parameters, and N and M correspond to the number of state variables and parameters. Here, we assume that the change of system parameters are significantly slow compared with those of the state variables, i.e., all parameter values do not change during the period T to calculate the maximum Lyapunov exponent.

At $t = t^*$, a point $\boldsymbol{x}^* \in \mathbb{R}^N$ satisfying

$$\boldsymbol{x}^* - \boldsymbol{f}(\boldsymbol{x}^*, \boldsymbol{p}(t^*)) = \boldsymbol{0} \tag{2}$$

becomes a fixed point of $\boldsymbol{f}$. We express the Jacobian matrix of $\boldsymbol{f}$ at $\boldsymbol{x}(t^*) = \boldsymbol{x}^*$ and $\boldsymbol{p}(t^*)$ as

$$D\boldsymbol{f}(\boldsymbol{x}^*, \boldsymbol{p}(t^*)) = \left. \frac{\partial}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{p}(t^*)) \right|_{\boldsymbol{x}=\boldsymbol{x}^*}. \tag{3}$$

The topological property of $\boldsymbol{x}^*$ is determined on the basis of the arrangement of all characteristic multipliers, μ_i $(i = 1, 2, \ldots, N)$, corresponding to the eigenvalues of $D\boldsymbol{f}(\boldsymbol{x}^*, \boldsymbol{p}(t^*))$. When one or more characteristic multipliers of $\boldsymbol{x}^*$ are on the circumference of a unit circle in the complex plane, a bifurcation occurs because of the change of its topological property. Kawakami [4] has classified the bifurcation types and has proposed a computational method of bifurcation points. Focusing on the stability of $\boldsymbol{x}^*$, it is stable when all the characteristic multipliers are in the unit circle; otherwise it is unstable. In the similar way as a fixed point, the topological property of a periodic point can be determined on the basis of characteristic multipliers, because a point $\boldsymbol{x}^*$ satisfying

$$\boldsymbol{x}^* - \boldsymbol{f}^n(\boldsymbol{x}^*, \boldsymbol{p}(t^*)) = \boldsymbol{0} \tag{4}$$

becomes an n-periodic point.

Let $\boldsymbol{x}(0)$ be an initial point. We choose an arbitrary point $\boldsymbol{w}(0) \in \mathbb{R}^N$ in the vicinity of $\boldsymbol{x}(0)$ and iterate as

$$\boldsymbol{v}(t) = \frac{\boldsymbol{w}(t)}{\|\boldsymbol{w}(t)\|}, \quad \boldsymbol{w}(t+1) = D\boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{p}) \cdot \boldsymbol{v}(t) \tag{5}$$

where $\|\cdot\|$ denotes the Euclidean norm of a vector. The maximum Lyapunov exponent of the sequence $\boldsymbol{x}(t)$ emanating from $\boldsymbol{x}(0)$ can be defined by

$$\lambda(\boldsymbol{x}(0), \boldsymbol{p}, T) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \ln \frac{\|D\boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{p}) \cdot \boldsymbol{v}(t)\|}{\|\boldsymbol{v}(0)\|}. \tag{6}$$

If the sequence $\boldsymbol{x}(t)$ emanating from $\boldsymbol{x}(0)$ converges into a stable fixed or periodic point, then the value of $\lambda(\boldsymbol{x}(0), \boldsymbol{p}, T)$ corresponds to $\ln(\mu_{\max})$ where

$$\mu_{\max} = \operatorname*{argmax}_i |\mu_i|. \tag{7}$$

Hence, the limit of sequence emanating from $\boldsymbol{x}(0)$ is a stable fixed or periodic point if $\lambda(\boldsymbol{x}(0), \boldsymbol{p}, T)$ takes a negative value. On the basis of the fact, we can predict the occurrence of a bifurcation of stable fixed and periodic points by observing the value of $\lambda(\boldsymbol{x}(0), \boldsymbol{p}, T)$.

3. Problem Statement and Controlling Method

Let us rewrite the map $\boldsymbol{f}$ in Eq. (1) as

$$\boldsymbol{x}(t+1) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{q}(t), \boldsymbol{r}(t)) \tag{8}$$

where $\boldsymbol{q}(t) \in \mathbb{R}^K$ denotes the vector of system parameters which can be changed by any environmental change of the system and the value is not adjustable; $\boldsymbol{r}(t) \in \mathbb{R}^L$ with $L = M - K$ represents the vector of adjustable system parameters.

In the system (8), we now consider the situation that a bifurcation of a fixed or periodic point can occur by changing the value of $\boldsymbol{q}(t)$. To simplify the problem, we also assume that $\boldsymbol{q}(t)$ and $\boldsymbol{r}(t)$ change every mT $(m = 1, 2, \ldots)$. Moreover, to simplify the following mathematical notations, we express $\lambda(\boldsymbol{x}(t_0), \boldsymbol{q}, \boldsymbol{r}, T)$ for $\boldsymbol{x}(t_0)$, $t_0 = mT$ $(m = 0, 1, 2, \ldots)$ as simply λ. On the basis of these preparations and assumptions, we propose a controlling method to keep the value of $\boldsymbol{r}(t)$ away from the bifurcation point when the values of $\boldsymbol{q}(t)$ and $\boldsymbol{r}(t)$ are in the vicinity of a bifurcation point as follows.

To solve the problem on avoiding bifurcations, we shall consider a minimization problem with respect to the objective function described by

$$J(\lambda) = \frac{1}{2}\left(\lambda - P(\lambda)\right)^2. \tag{9}$$

Here, we define the projection P as

$$P(\lambda) = \begin{cases} \lambda & \text{if } \lambda \le \lambda^* \\ \lambda^* & \text{otherwise} \end{cases} \tag{10}$$

where $\lambda^* < 0$ and its value is set up by a user in order to detect the approach of system-parameter values to a bifurcation point of a fixed or periodic point. Hence, we presume that system parameters are close to a bifurcation point when $\lambda^* < \lambda < 0$.

To avoid the occurrence of a bifurcation, we introduce the updating rule of $\boldsymbol{r}(t)$ on the basis of the gradient system

$$\boldsymbol{r}((m+1)T) - \boldsymbol{r}(mT) = -\gamma \frac{\partial J(\lambda)}{\partial \boldsymbol{r}} = -\gamma(\lambda - \lambda^*)\frac{\partial \lambda}{\partial \boldsymbol{r}}. \tag{11}$$

Let r_ℓ be the ℓth entry of $\boldsymbol{r}$. $\partial\lambda/\partial r_\ell$ that is the ℓth entry of $\partial\lambda/\partial\boldsymbol{r}$ in Eq. (11) is derived as follows. From Eqs. (5) and (6), λ for $\boldsymbol{x}(t_0)$ can be calculated as

$$\lambda = \lim_{T\to\infty} \frac{1}{T} \sum_{t=mT}^{(m+1)T-1} \log\|\boldsymbol{w}(t+1)\|. \tag{12}$$

By calculating the partial differentiation of Eq. (12) with r_ℓ, we obtain

$$\frac{\partial\lambda}{\partial r_\ell} = \frac{1}{T} \sum_{t=mT}^{(m+1)T-1} \frac{\boldsymbol{w}(t+1)^\top}{\|\boldsymbol{w}(t+1)\|^2} \cdot \frac{\partial \boldsymbol{w}(t+1)}{\partial r_\ell} \tag{13}$$

$$\frac{\partial\|\boldsymbol{w}(t+1)\|}{\partial r_\ell} = \frac{\boldsymbol{w}(t+1)^\top}{\|\boldsymbol{w}(t+1)\|} \cdot \frac{\partial \boldsymbol{w}(t+1)}{\partial r_\ell} \tag{14}$$

where $\top$ denotes the transpose of a vector.

Since $\boldsymbol{q}(t)$ and $\boldsymbol{r}(t)$ are changed at $t = mT$, we have the partial differentiation terms in the right hand side of Eqs. (13) and (14) as

$$\frac{\partial \boldsymbol{w}(t+1)}{\partial r_\ell} = \frac{\partial D\boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{q}, \boldsymbol{r})}{\partial r_\ell} \cdot \boldsymbol{v}(t) + D\boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{q}, \boldsymbol{r}) \cdot \frac{\partial \boldsymbol{v}(t)}{\partial r_\ell}. \tag{15}$$

Moreover, the partial differentiation at the first term in the right hand side of Eq. (15) can be calculated as

$$\frac{\partial D\boldsymbol{f}(\boldsymbol{x}(t),\boldsymbol{q},\boldsymbol{r}))_{ij}}{\partial r_\ell} = \sum_{k=1}^{N}\left(\frac{\partial^2 f_i(\boldsymbol{x}(t),\boldsymbol{q},\boldsymbol{r})}{\partial x_j \partial x_k}\cdot\frac{\partial x_k(t)}{\partial r_\ell}\right) + \frac{\partial^2 f_i(\boldsymbol{x}(t),\boldsymbol{q},\boldsymbol{r})}{\partial x_j \partial r_\ell} \tag{16}$$

where $D\boldsymbol{f}(\cdot)_{ij}$ denotes the (i,j)th entry of the Jacobian matrix and $\partial x_k(t)/\partial r_\ell$ can be calculated from the first variational equation of $\boldsymbol{x}(t)$ with respect to r_ℓ defined by

$$\frac{\partial \boldsymbol{x}(t+1)}{\partial r_\ell} = D\boldsymbol{f}(\boldsymbol{x}(t),\boldsymbol{q},\boldsymbol{r})\cdot\frac{\partial \boldsymbol{x}(t)}{\partial r_\ell} + \frac{\partial \boldsymbol{f}(\boldsymbol{x}(t),\boldsymbol{q},\boldsymbol{r})}{\partial r_\ell}. \tag{17}$$

The partial differentiation at the second term in the right hand side of Eq. (15) becomes

$$\frac{\partial \boldsymbol{v}(t)}{\partial r_\ell} = \frac{1}{\|\boldsymbol{w}(t)\|}\cdot\frac{\partial \boldsymbol{w}(t)}{\partial r_\ell} - \frac{1}{\|\boldsymbol{w}(t)\|^2}\cdot\frac{\partial\|\boldsymbol{w}(t)\|}{\partial r_\ell}\cdot\boldsymbol{w}(t). \tag{18}$$

In Eqs. (13)–(18), we can set to $\partial\boldsymbol{x}(0)/\partial r_\ell = \mathbf{0}$, $\partial\boldsymbol{v}(0)/\partial r_\ell = \mathbf{0}$, $\partial\boldsymbol{w}(0)/\partial r_\ell = \mathbf{0}$.

4. Experimental Results and Discussion

To show the validity of the proposed method, we carried out experiments on avoiding the bifurcations of fixed and periodic points observed in the Hénon map defined by

$$x_1(t+1) = 1 + x_2(t) - q(t)\cdot x_1(t)^2 \tag{19a}$$
$$x_2(t+1) = r(t)\cdot x_1(t) \tag{19b}$$

where $\boldsymbol{x} = (x_1, x_2)^\top$ is the vector of state variables, and $q(t)$ and $r(t)$ are time-variant system parameters.

Before experiments, we found bifurcation points of fixed and periodic points observed in Eq. (19) using the method of bifurcation analysis [4]. Figure 1 plots their bifurcation curves on the plane of the system parameters. In this bifurcation diagram, the black solid curve indicated by I^n denotes the period-doubling bifurcation set of n-periodic point; the curve indicated by I^1 corresponds to the period-doubling bifurcation set of the fixed point. The stable fixed point exists in the left-hand-side parameter region of the curve I^1 and the stable n-periodic point can be observed in the parameter area surrounded with the two curves $I^{n/2}$ and I^n. We explain the meanings of the red and blue curves and the black and white circles later.

Let us demonstrate the proposed controller can avoid bifurcations of fixed and periodic points. In our experiments, we assumed that the values of $q(t)$ and $r(t)$ are non-adjustable and adjustable, respectively. Hence, we considered the situation that $q(t)$ was changed by any reason. The value of $r(t)$ was updated on the basis of Eqs. (11)–(18) so that a bifurcation of a stable fixed or periodic point never appear. In Eqs. (11) and (12), we set as $\gamma = 0.1$, $\lambda^* = -0.1$, and $T = 500$.

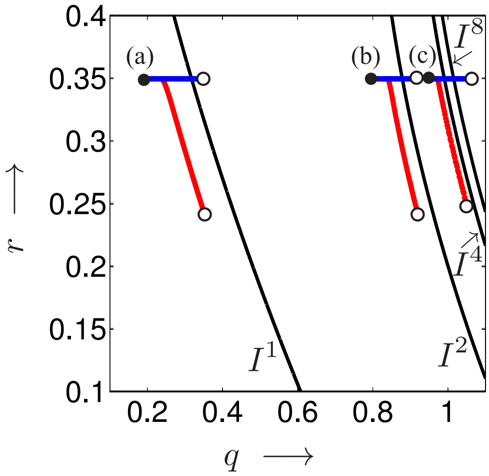

Figure 1: Bifurcation diagram and experimental results

First, we set the initial values as $q(0) = 0.2$, $r(0) = 0.35$, and $\boldsymbol{x}(0) = (1.2, -0.1)^\top$ so that it converges to the fixed point. The values of $q(0)$ and $r(0)$ correspond to the black point (a) in Fig. 1. Let us consider the situation that $q(t)$ gradually increases every the period T and then crosses the bifurcation curve I^1. In Fig. 1, the blue line emanating from the black (initial) point denotes the locus of parameter changes in the simulation without our controller; the red curve ramified from the blue line expresses those with our controller. The white circles at the end of blue line and red curve express the end of simulation. Hence, the fixed point bifurcates without our controller because the blue line crosses the I^1 curve. On the other hand, as seen from the locus of the red curve, the proposed controller avoided the bifurcation.

Figure 2(a) shows the loci of x_1, λ, q, and r as time series. The blue and red dots correspond to “without control” and “with control” as Fig. 1. In the case without control, the value of λ with blue dots gradually increased along the passage of time, and then the stable fixed point bifurcated at around $t = 80T$ because λ reached zero. However, as seen the red dots, after $t = 24T$, the value of λ was inhibited to approximately λ^* by controlling $r(t)$ with the proposed method. As the results, the period-doubling bifurcation of the fixed point did not occur.

Second, we treated the problem on avoiding the period-doubling bifurcation of the stable two-periodic point. Here, we set $q(0) = 0.8$, $r(0) = 0.35$, and $\boldsymbol{x}(0) = (1.3, -0.2)^\top$ so that the stable two-periodic point appeared. This initial parameter setting corresponds to the black point (b) in Fig. 1. We also defined the increase of $q(t)$ so that it could cross the bifurcation curve I^2 as shown in the blue locus starting from the black point (b) shown in Fig. 1. For the situation, our controller inhibited the value of λ so that

$\lambda < \lambda^*$. The inhibition was observed as the red loci of λ in Fig. 2(b), the updating $r(t)$ with our controller was begun at $t = 33T$ such that $\lambda > \lambda^*$. As the results, the two-periodic point was kept without bifurcating at about $t = 65T$.

Finally, we also experimented on avoiding the period-doubling bifurcation of a four-periodic point. The initial values were set as $q(0) = 0.95$, $r(0) = 0.35$, and $\boldsymbol{x}(0) = (1.0, -0.25)^\top$. In this setting corresponding to the black point (c) in Fig. 1, the stable four-periodic point could be observed. Since we also assumed that $q(t)$ could cross the two period-doubling bifurcations, I^4 and I^8, in the case without control, the chaotic state was observed after about $t = 72T$ in Fig. 2(c). In contrast, the proposed controller worked well so that the bifurcations were avoided as seen in Figs. 1 and 2(c).

5. Conclusion

In this paper, we dealt with the problem avoiding bifurcations of a stable fixed or periodic point observed in a discrete-time dynamical system. We proposed a novel method derived from a minimization problem with respect to the maximum Lyapunov exponent. We also demonstrated that the proposed method is valid through experiments to avoid bifurcations of fixed and periodic points observed in the Hénon map.

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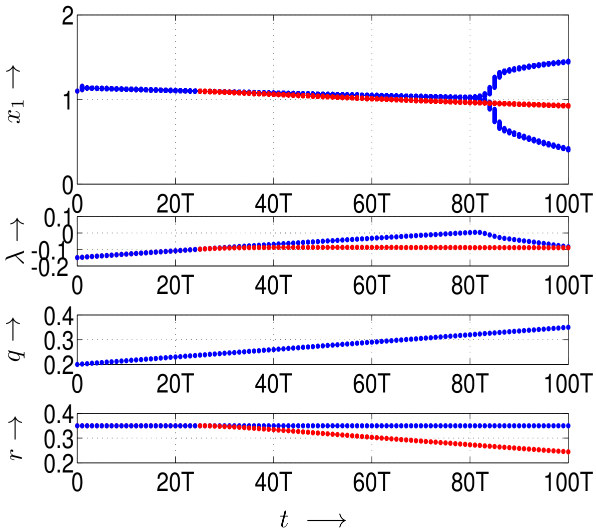
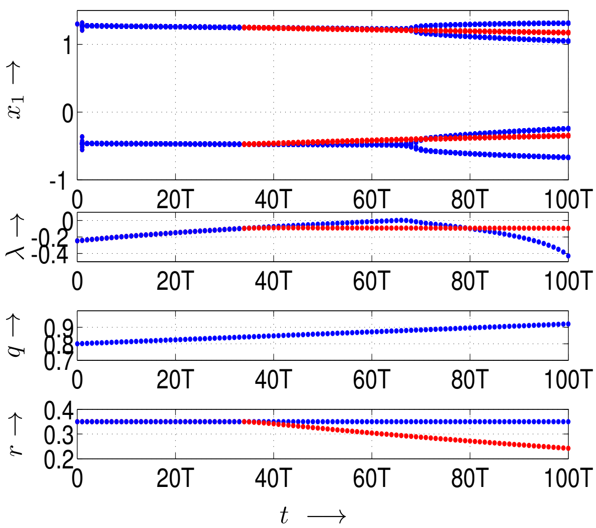
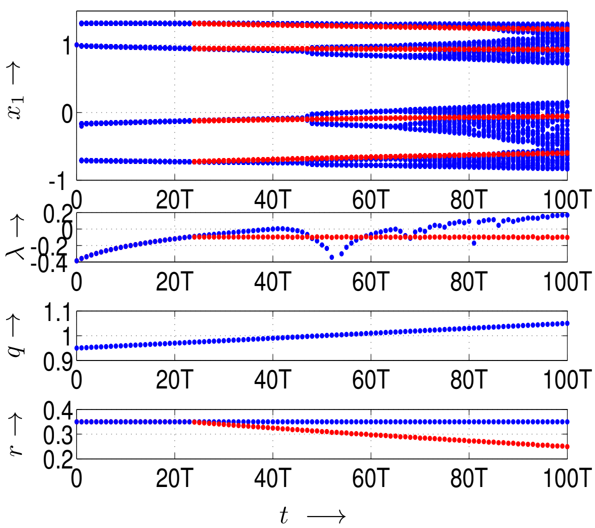


(a) fixed point



(b) two-periodic point



(c) four-periodic point

Figure 2: Experimental results displayed as time series