

Performance Evaluation of the Factor Graph CPM Phase Discriminator Decoder with Canonical Messages and Modulo Mean Updates

Jan Sykora

Czech Technical University in Prague
E-mail: jan.sykora@fel.cvut.cz

Pavel Prochazka

Czech Technical University in Prague
E-mail: prochp6@fel.cvut.cz

The paper addresses the CPM class of the modulation passing through the parametric channel with the factor graph decoding inherently coping with the channel parameter. The combination of the discrete channel symbols with continuous valued channel parameter leads generally to complex mixture densities difficult to cope in the sum-product algorithm message updates in the factor graph. We attack the problem by, first, performing the factor graph operations in phase-space domain, and, second, by defining proper canonical densities described by a limited set of parameters. We derive proper factor and variable node update rules respecting the modulo 2π nature of the phase operations. We evaluate the error rate performance and compare it against the brute-force marginalization rules.

I. INTRODUCTION

A. Background

The Continuous Phase Modulation (CPM) has excellent resistance to the nonlinear distortion due to its constant envelope property. The CPM modulator contains the recursive memory block—continuous phase encoder [1]. This makes the CPM a very good inner component for the serially concatenated encoding schemes [2].

The Factor Graphs (FG) and the Sum-Product Algorithm (SPA) ([3], [4]) is a general framework for evaluation of the decision measure in variety of the system setups including all forms of the concatenated component codes and modulators (e.g. [5], [6]). The FG/SPA technique can be also used to inherently cope with the parametric channel introducing nuisance channel parameters of various forms (phase, channel transfer, etc.) [7], [8], [9], [10]. One of the major problems arising from the inclusion of continuous valued channel parameters is the fact that the SPA becomes practically difficult to implement. The SPA defines the marginalization update rules for the factor node output messages (see [3], [4]) where the continuous parameters with general densities introduce the necessity to integrate when calculating the factor node marginalization. This is very difficult and complex to be directly implemented. Several solutions (approximations) solving this problem appeared. They are centered around the idea of using

the approximation of the densities by the restricted class of *parameterizable* canonical distributions [11], [12]. The marginalization factor node update thus evaluates only the new set of the density parameters.

The constant envelope property allows to use a Limiter Phase Discriminator (LPD) receiver front end processing. It is generally suboptimal (see e.g. [13] for the information efficiency in a broader context of the receiver with nonlinear projection). However, the LPD front end simplifies the receiver substantially by reducing the dimensionality of the processing (the signal space representation of the CPM is a multidimensional one). The phase-space output of the LPD is single-dimensional. The LPD also removes the need for the amplitude estimator (or automatic gain control) at the receiver.

B. The Contribution of the Paper

The contribution of the paper is threefold. (1) The work in [14] is extended by introducing the new canonical message type—the Circularly Expanded Gaussian Message. It improves the precision of the modulo mean update equations for the single component messages. (2) We introduced the principle of the message erasure in the situation of the ambiguity resolution decision not having strong enough input information for doing that decision. This leads to the lower probability of making a wrong ambiguity decision. (3) We have evaluated the bit error rate performance and discussed the behaviour of the decoder for the variety of the system parameters and compared that to the “brute-force” solution using the direct integration of the sampled message densities.

II. SYSTEM MODEL AND DEFINITIONS

We consider a message of binary data symbols $\mathbf{b} = [\dots, b_n, \dots]^T$ encoded by the outer encoder into the codeword $\mathbf{c} = [\dots, c_n, \dots]^T$ passing through the interleaver Π having the interleaved codeword $\mathbf{d} = [\dots, d_n, \dots]^T$ at its output. This forms an input of the CPM modulator. It is modeled using the Continuous Phase Encoder (CPE) with output M_q -ary symbols $q_n \in \{0, \dots, M_q - 1\}$ and the Nonlinear Memoryless Modulator (NMM) [1] with output signal $s(t) = \exp(j\phi(t))$, where

$$\phi(t) = \kappa \sum_n \beta(q_n, t - nT_S). \quad (1)$$

The κ is the modulation index, β is the nonlinear phase function of the NMM and T_S is the symbol period. The

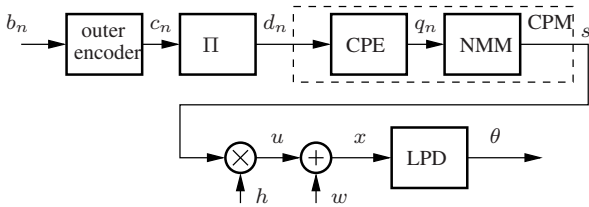


Figure 1. The system model.

channel transfer $h = \exp(j\varphi)$ is assumed to have the random phase rotation φ with uniform distribution. The AWGN is $w(t)$. The received signal $x(t)$ passes through the *nonlinear* Limiter Phase Discriminator (LPD) with the output $\theta = \angle(x)$. See Fig. 1.

III. FACTOR GRAPH CPM PHASE DISCRIMINATOR ITERATIVE DECODER

A. Sampled Phase Space FG Iterative Decoder

The LPD receiver front end is an elegant, however generally slightly suboptimal (see [13] for the capacity degradation evaluation) solution of the CPM demodulator/decoder. It overcomes the necessity of the multidimensional Euclidean constellation space processing and also lifts the need for the channel transfer amplitude estimator. This is particularly useful in a mobile fading environment with short message frames. See [14] for a detailed discussion on the FG receiver processing options. The sampled phase space FG processing is shown on Fig. 2. The factor nodes f_ϕ describe relation between the channel symbol q_n and the corresponding sample $\phi_{n,m}$ (m th sample within n th symbol period, N_S samples per one symbol) of the modulated signal phase $\phi(t)$.

B. Sampled PDF Messages & Rectangular Rule Update Approximation

In order to make a performance comparison for the modulo mean canonical messages, we need a *reference case*. The ideal would be the one with straight evaluation of the marginalization integrals over the mixture continuous/discrete PDFs. The messages would be an arbitrary continuous valued functions on the $[0, 2\pi)$ support. Such system is impossible to implement. A very good approximation is a numerical approximation of the marginalization integrals done by the sampling the PDF and using a rectangle rule. The number of the rectangular rule intervals over the $[0, 2\pi)$ support is denoted by N_f . The approximation is however useful *only as a simulation reference scenario*. A practical implementation would suffer from a high complexity due to the necessity to pass large set of the sampled values and also due to the need of the high volume summation (the integral approximation). This FG message representation will be called Sampled Density Messages (SDM).

C. Discrete Messages

The discrete message type is associated with variable node (VN) q_n (or equivalently with $\phi_{n,m}$). The backward message (a priori probability mass function) is $\mathcal{M}_B\{q_n\} = \{p(q^{(i)})\}_{i=0}^{M_q-1}$ and the forward message (a posteriori probability for a given observation) is

$\mathcal{M}_F\{q_n\} = \{p(\theta_n^0|q^{(i)})\}_{i=0}^{M_q-1}$. The decision measure is $\mathcal{M}\{q_n\} = \mathcal{M}_F\{q_n\}\mathcal{M}_B\{q_n\}$. A generic message, that can become any of $\mathcal{M}\{q_n\}, \mathcal{M}_F\{q_n\}, \mathcal{M}_B\{q_n\}$, will be denoted by $\mu\{q_n\}$.

D. Canonical Messages

We use a class of canonical messages as an approximation to the true densities of the PDFs associated with the messages. The canonical message is fully represented by a relatively small set of the parameters that can be passed from the node to the node. Also the message update rules operating on the canonical densities are typically a simple manipulations comprising relatively small set of input and output parameters. Compare this with the theoretical case of the arbitrary shape continuous PDF or its approximation by the sampled PDF. The former is not directly implementable in the digital signal processing hardware. The latter is implementable however it requires typically a large set of the parameters (PDF samples) to be passed as the messages. We define two forms of the canonical messages: Mixture Gaussian Message and Circularly Expanded Gaussian Message.

1) *Mixture Gaussian Messages*: The Mixture Gaussian Message (MGM) ([11], [12]) is the approximation of the true density by the weighted superposition of the shifted Gaussian PDFs. The MGM models quite well the the marginalized output PDF when the discrete and Gaussian density are jointly at the input of the factor node. The circular $\text{mod}2\pi$ nature of the phase operations are handled only through the modulo mean message update rules at the factor node.

The MGMs are formed by parametric class of the densities

$$p(x) = \sum_{i=1}^{M_x} p(x^{(i)})p_{\mathcal{N}(\sigma_x^2)}(x - x^{(i)}) \quad (2)$$

where $p_{\mathcal{N}(\sigma_x^2)}(x)$ is the PDF of *zero-mean, σ_x^2 variance real-valued* Gaussian random variable, and $p(x^{(i)})$ is the probability of the i th component. The *generic* messages are

$$\mu\{x\} = \left\{ \{p(x^{(i)})\}_{i=1}^{M_x}, \{x^{(i)}\}_{i=1}^{M_x}, \sigma_x^2 \right\}, \quad x^{(i)} \in [0, 2\pi). \quad (3)$$

2) *Circularly Expanded Gaussian Messages*: The Circularly Expanded Gaussian Message (CEGM) solves the problem of the MGM with the *single component only*. In a situation with the single component being close to the 0 or 2π phase, the MGM (even with the modulo mean message update rules) can result in a biased resulting output PDF.

The CEGM approximation is attacking this problem by superposing three mutually 2π shifted versions of the MGM single component message (Fig. 3). The resulting PDF, with the proper $\text{mod}2\pi$ update operations, models correctly the circular nature of the PDF. The circularly expanded PDF is properly scaled to have a unity area.

The CEGMs are formed by

$$p(x) = \frac{1}{3} \sum_{m=-1}^1 p_{\mathcal{N}(\sigma_x^2)}(x - x^{(1)} - m2\pi). \quad (4)$$

The generic messages are the same as for the *single component MGM* $\mu\{x\} = \{x^{(1)}, \sigma_x^2\}$, $x^{(1)} \in [0, 2\pi)$.

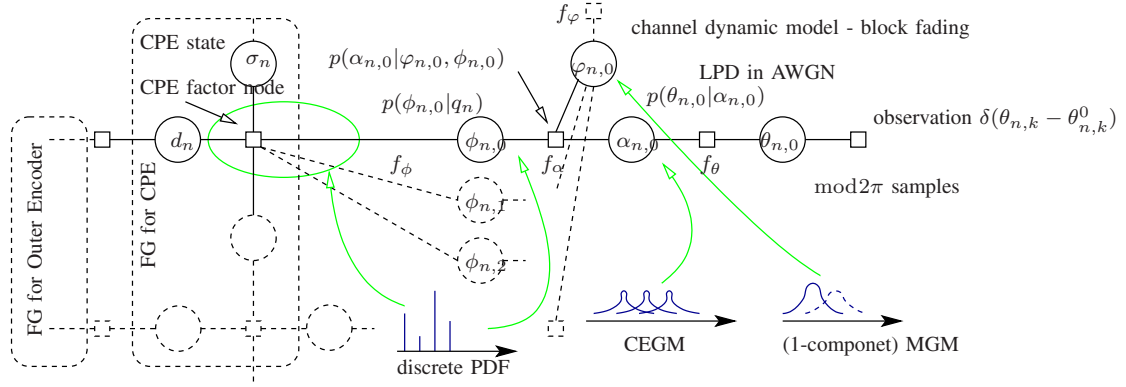


Figure 2. The Sampled Phase Space FG model.

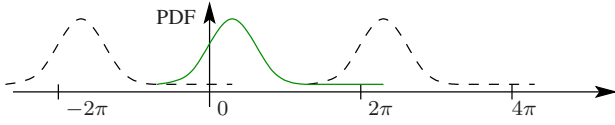


Figure 3. The Circularly Expanded Gaussian Message PDF.

E. Modulo Mean Update Rules

Here, we extend the results derived in [14]. Particularly, the erasure principle is applied to the ambiguity resolution and also the f_α factor node equations are amended for the CEGM. Some parts of the derivation are skipped for the brevity, see details in [14].

All operations in Modulo Mean Update Rules (MMUR) are performed with $\text{mod } 2\pi$ restriction *only* applied on the set of *component mean values* $\{x_i\}_i$. The tails of the component Gaussian densities are allowed to go outside the $[0, 2\pi)$ range. The support of all involved PDFs is the complete real axis, however the main probability mass lies within the $[0, 2\pi)$ range. This *approximates* the circular nature of the true $\text{mod } 2\pi$ arithmetics but keeps the composite densities *Gaussian* and therefore *easily tractable*. The MMUR will be applied to both, the MGMs and the CEGMs.

F. (q, w, x) Factor Node $x = (q + w) \text{ mod } 2\pi$

This factor node class represents factor node f_α with $x = \alpha$, $w = \varphi$. The associated factor is $p(x|q, w) = \delta(x - (q + w) \text{ mod } 2\pi)$. The message $\mu\{w\}$ is assumed to be MGM with *single component*, i.e. $M_w = 1$. The message $\mu\{q\}$ is assumed to be a *discrete* one, i.e. $\sigma_q^2 \rightarrow 0$. The message $\mu\{x\}$ is assumed to be the CEGM.

1) $\mu\{x\}$ message: Since the $\mu\{x\}$ message is on the open branch of the FG leading to the observation, it does not need to be evaluated.

2) $\mu\{w\}$ message: The assumption of a single component $\mu\{w\}$ message requires the *ambiguity resolution*. We adopt a simple *ambiguity resolution strategy* based on selecting the most probable components in the messages $\mu\{q\}$ and $\mu\{x\}$. This is however reliable only when the $\max_j p(x^{(j)})$ and $\max_i p(q^{(i)})$ are sufficiently separable from the remaining values. If this does not hold, we apply the *erasure principle*, i.e. we rather select the non-informing density than making the wrong decision. The erasure is achieved by assigning $\mu\{w\} = \text{void}$. If this

value enters any update calculation for the remaining FG edges, it forces to produce the uniform distribution (e.g. on the $\mu\{q\}$). The decision on the erasure is done when the metric gets below the threshold

$$\rho(\{p(q^{(i)})\}_i) = p_{M1} - p_{M2} < \rho_{Th} \quad (5)$$

where p_{M1} and p_{M2} are the maximum and the second maximum value of the set $\{p(q^{(i)})\}_i$.

If there is no erasure, the update rule is

$$p(w^{(1)}) = 1, \quad (6)$$

$$w^{(1)} = \left(x^{(\arg \max_j p(x^{(j)}))} - q^{(\arg \max_i p(q^{(i)}))} \right) \text{ mod } 2\pi, \quad (7)$$

$$\sigma_w^2 = \sigma_x^2. \quad (8)$$

3) $\mu\{q\}$ message: The update rule for is

$$p(q^{(i)}) = \frac{\sum_j p(x^{(j)}) \sum_{m=-1}^1 e^{-\frac{(x^{(j)} - (q^{(i)} + w^{(1)}) \text{ mod } 2\pi - m2\pi)^2}{2(\sigma_x^2 + \sigma_w^2)}}}{3\sqrt{2\pi(\sigma_x^2 + \sigma_w^2)}}. \quad (9)$$

The set of values $\{q^{(i)}\}_i$ is defined a priori. The variance for the discrete message type is $\sigma_q^2 \rightarrow 0$.

G. (x, y) Factor Node $y = (x + w) \text{ mod } 2\pi$

We assume Factor Node (FN) with 2 edges. The x edge is CEGM $\mu\{x\}$. The y edge is assumed to be connected to the *observation* VN. Therefore its input message is a 1-component MGM message $\mu\{y\}$ with *Dirac delta PDF*. The associated operation is $y = (x + w) \text{ mod } 2\pi$ where w is zero-mean Gaussian noise with the variance σ_w^2 . The factor is $p(y|x) = p_{\mathcal{N}(\sigma_w^2)}(y - x)$. This FN is applicable to f_θ (Fig. 2).

1) $\mu\{y\}$ message: The $\mu\{y\}$ message does not need to be evaluated since it leads to the open branch connected to the observation.

2) $\mu\{x\}$ message: The update rules are

$$p(x^{(1)}) = 1, p(x^{(i)}) = 0, i \neq 1; x^{(i)} = y^{(i)}; \sigma_x^2 = \sigma_w^2. \quad (10)$$

H. (w_1, w_2, w_3) Variable Node for 1-component MGM

We assume VN with 3 connected edges with the messages $\mu\{w_1\}, \mu\{w_2\}, \mu\{w_3\}$ (the VN is completely symmetric). All messages are assumed to be 1-component MGM ones. This is applicable to the VN $\varphi_{n,k}$ in Fig. 2.

The update equations can be derived as

$$w_3^{(1)} = \frac{w_1^{(1)} \sigma_{w_2}^2 + w_2^{(1)} \sigma_{w_1}^2}{\sigma_{w_2}^2 + \sigma_{w_1}^2}; \sigma_{w_3}^2 = \frac{\sigma_{w_2}^2 \sigma_{w_1}^2}{\sigma_{w_2}^2 + \sigma_{w_1}^2}. \quad (11)$$

IV. PERFORMANCE EVALUATION

A. System setup

We have evaluated the FG iterative decoder performance by means of the computer simulation. The goal was to compare the performance of the FG with MGM and CEGM with MMUR in comparison with the “brute-force” solution using a direct integration of the sampled densities. The expectation is that the MGM/CEGM/MMUR approximations somewhat deteriorate the performance w.r.t. the SDM. The implementation complexity of the update equations and the number of the parameters needed to describe the messages are however much lower than for the SDMs.

The system used in the simulation was the uncoded binary CPM with rectangular (REC1) frequency pulse and $\kappa = 1/2$, i.e. MSK. We used very short bursts with ambiguity resolution preamble of the length $T_p = 5$ symbols. The preamble is implemented by setting the a priori PDF of the corresponding message in FG to the Dirac delta at given known symbol. The overall frame length was T symbols. Notice that the uncoded system has FG with very short loops which strongly affects the convergence properties. The number of samples per symbol is $N_S = 2$. We used random channel phase φ_n model where the phase was assumed being constant during the frame duration. The signal-to-noise ration is defined as $\gamma = \bar{\mathcal{E}}_S / N_0$ where $\bar{\mathcal{E}}_S$ is the mean symbol energy and N_0 is the AWGN noise power spectrum density.

The uncoded system used in this paper is a first step to the main target system, i.e. the serially concatenated coded system with the CPM as the inner component. We needed a system simple enough to verify the concept. As a simple system, it, of course, suffers from number of deficiencies—all FG loops are very short, there are no tail flush bits at the end of the frame and also the channel model is a very simple one. All those limitations will be addressed in the future work.

The SDM FG uses $N_f = 8$ (or 16) phase samples per $[0, 2\pi)$ interval. We can also get the channel phase estimates as a side effect of the simulation in both, MGM/CEGM/MMUR and SDM FG models. It is important to stress that the receiver FG processing handles the channel parameter inherently inside the iterative algorithm (there is no separate synchronizer). We can however evaluate the inherent phase estimation value as the the maximizer of the corresponding decision measure associated with the φ_n variable in the FG $\hat{\varphi}_n = \arg \max_{\varphi} \mathcal{M}\{\varphi_n\}$.

The numerical simulation results are shown in Fig. 4, 5. We have evaluated several different sets of the system setups in order to identify performance bottlenecks. There are variants with the restricted and unrestricted channel

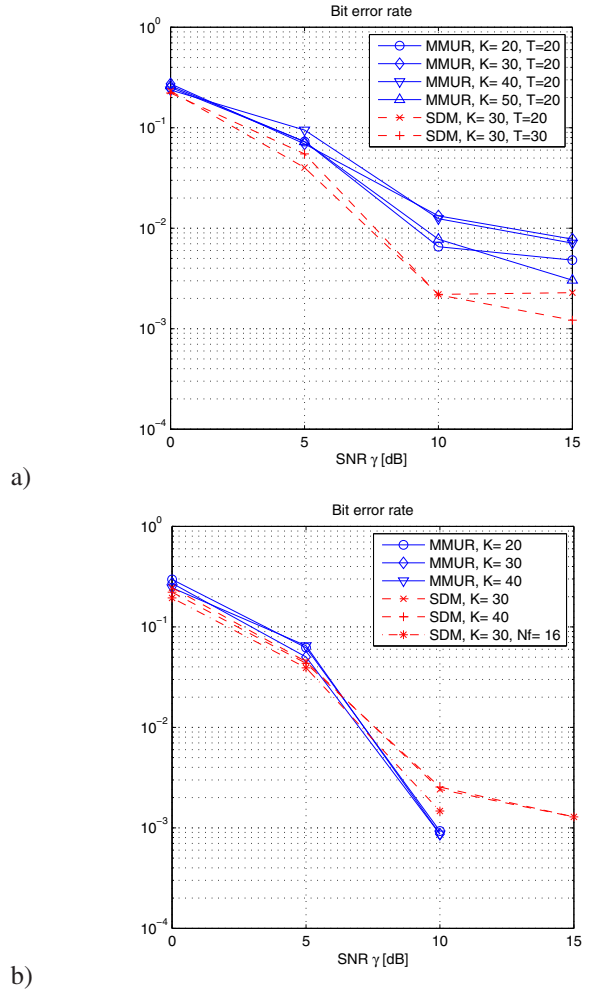


Figure 4. Bit error rate for a) unrestricted, b) restricted phase.

phase. The unrestricted channel phase is a per-frame constant random variable on the full range $[0, 2\pi)$. The restricted phase is again a random variable but it avoids the close vicinity of the 0 or 2π values to test the impact of the MGM canonical density behaviour around those points. We have also separated the frames where the ambiguity resolution (supported by the pilot symbols) failed. Those frames are excluded from the BER measurement. The graphs of the phase estimation errors shows clearly when this happens. The simulations revealed several phenomena which are discussed in the following.

B. Ambiguity resolution failures

The SDM system exhibits visibly lower ambiguity failure rate (see Fig. 5) but on the other side the stationary phase estimation error (mean square error value) is higher under the same conditions on signal-to-noise ration, and number of iterations. The higher susceptibility of the MMUR to ambiguity errors is apparently caused by the asymmetric behaviour of the MGM associated with the phase φ_n node when the phase is close to the $\text{mod}2\pi$ boundaries. This message apparently forces the mean in a wrong direction not informing correctly about the other likely possibility at the other boundary.

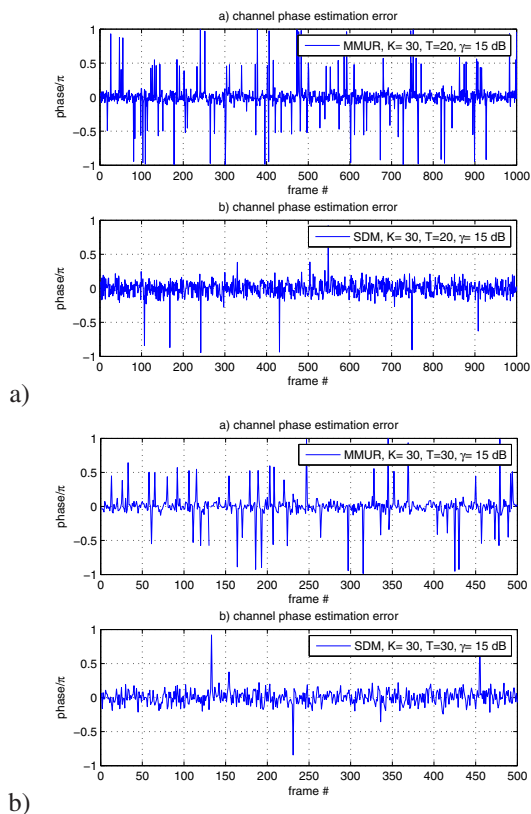


Figure 5. Phase estimation error for a) unrestricted, b) restricted phase.

C. Channel phase in the vicinity of the $\text{mod}2\pi$ boundaries

The influence of the MGM (associated with the phase φ_n) message asymmetry in the vicinity of the $\text{mod}2\pi$ boundaries is a likely cause for the error floor in the case of the unrestricted channel phase. Compare Fig. 4 a) and b). For the restricted phase, the problem is intentionally eliminated by avoiding those phase values. As a consequence, the error floor disappears. For large signal-to-noise ratios ($\gamma = 15$ [dB]), the number of errors (missing points in the graph) was zero for given size of the simulation runs.

It is apparent that the future improvements of the MMUR will have to include the *circular expansion on all mixture messages* in a similar way as it is used in limited parts of the current FG by employing the CEGM.

D. Error floor

The SDM error performance exhibits the error floor regardless of the restriction of the phase range. The likely cause for this behavior is the granularity of the rectangular rule approximation. There is clear limit of the message information precision given by the number of sampling points. On the other side, the MMUR with MGM/CEGM does not suffer by this. The particular mean and variance of the individual message component can attain an arbitrary precise value. This becomes particularly important for higher signal-to-noise ratios.

V. CONCLUSIONS

We have extended our initial work on the FG framework for the LPD CPM receiver by several new concepts—namely the circular expansion of the message

CEGM and the erasure concept for non-informing input messages. We have compared the performance of the MGM/CEGM/MMUR FG system with the FG directly (through rectangular approximation) applying the marginalization update rules. These two systems each exhibit their own deficiencies. The *SDM is critically dependent on the granularity* of the rectangular rule approximation. It is important mainly at high signal-to-noise ratios. Also, in order to achieve fine granularity, the number of parameters representing the messages is very high and it also leads to computationally demanding evaluation. On the other side, the MMUR has a *constant finite and small set* of the canonical density parameters that can represent the message with *high precision* even at high signal-to-noise ratio. The MMUR with MGM/CEGM outperforms the SDM (with a similar size of the parameter set) system at high signal-to-noise ratio. The weak side of the canonical messages in the MGM form is their susceptibility on the vicinity of the $\text{mod}2\pi$ boundaries. It is removed by using CEGM. However, at the current stage, we were not able to use the CEGM through the whole FG due to the limitations of the update rules derived so far only for MGM. This deficiency will be addressed by the future work.

REFERENCES

- [1] B. E. Rimoldi, "A decomposition approach to CPM," *IEEE Trans. Inf. Theory*, vol. 34, pp. 260–270, Mar. 1988.
- [2] P. Moqvist and T. M. Aulin, "Serially concatenated continuous phase modulation with iterative decoding," *IEEE Trans. Commun.*, vol. 49, pp. 1901–1915, Nov. 2001.
- [3] F. R. Kschischang, B. J. Frey, and H.-A. Loeliger, "Factor graphs and the sum-product algorithm," *IEEE Trans. Inf. Theory*, vol. 47, pp. 498–519, Feb. 2001.
- [4] H.-A. Loeliger, "An introduction to factor graphs," *IEEE Signal Process. Mag.*, vol. 21, pp. 28–41, Jan. 2004.
- [5] S. Benedetto, D. Divsalar, G. Montorsi, and F. Pollara, "A soft-input soft-output APP module for iterative decoding of concatenated codes," *IEEE Commun. Lett.*, vol. 1, pp. 22–24, Jan. 1997.
- [6] S. Benedetto, D. Divsalar, G. Montorsi, and F. Pollara, "Serial concatenation of interleaved codes: Performance analysis, design, and iterative decoding," *IEEE Trans. Inf. Theory*, vol. 44, pp. 909–926, May 1998.
- [7] A. Barbieri, G. Colavolpe, and G. Caire, "Joint iterative detection and decoding in the presence of phase noise and frequency offset," in *Proc. IEEE Internat. Conf. on Commun. (ICC)*, 2005.
- [8] A. Barbieri and G. Colavolpe, "Soft-output detection of differentially encoded M-PSK over channels with phase noise," in *Proc. European Signal Processing Conf. (EUSIPCO)*, 2006.
- [9] N. Niu, M. Shen, J. A. Ritcey, and H. Liu, "A factor graph approach to iterative channel estimation and LDPC decoding over fading channels," *IEEE Trans. Wireless Commun.*, vol. 4, pp. 1345–1350, July 2005.
- [10] F. Simoens and M. Moeneclaey, "Code-aided estimation and detection on time-varying correlated MIMO channels: A factor graph approach," *EURASIP J. on Advances in Signal Processing*, vol. 2006, pp. 1–11, 2006.
- [11] A. P. Worthen and W. E. Stark, "Unified design of iterative receivers using factor graphs," *IEEE Trans. Inf. Theory*, vol. 47, pp. 843–849, Feb. 2001.
- [12] G. Colavolpe, A. Barbieri, and G. Caire, "Algorithms for iterative decoding in the presence of strong phase noise," *IEEE J. Sel. Areas Commun.*, vol. 23, pp. 1748–1757, Sept. 2005.
- [13] J. Sykora, "Information waveform manifold based preprocessing for nonlinear multichannel modulation in MIMO channel," in *Proc. IEEE Global Telecommunications Conf. (GlobeCom)*, (St. Louis, USA), pp. 1–6, Nov. 2005.
- [14] J. Sykora, "Factor graph framework for serially concatenated coded CPM with limiter phase discriminator receiver," in *Proc. IEEE Vehicular Technology Conf. (VTC)*, (Baltimore, USA), pp. 1–6, Oct. 2007.