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Downlink Power Allocation for Wireless Systems through Network Utility Maximization

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***Abstract*— We study power allocation problems in wireless systems through network utility maximization framework. In this framework, the type of utility function represents the characteristics of each service. Hence, to deal with multi-class services in the system, it is important to accommodate various types of utility functions. In this paper, we develop a simple algorithm that can accommodate four types of utility functions in a unified way, while providing asymptotically optimal power allocation. Those types of utility functions can represent most of services in wireless networks. We first develop an algorithm that solves a general optimization problem and, then, show that our algorithm can be applied to the downlink power allocation problem.**

I. Introduction

Recently, network utility maximization framework has been extensively studied in network resource allocation problems for both wireless and wireline networks [1], [2], [3], [4], [5]. In this framework, each user has its utility function, which is a function of its resource allocation and resource is allocated so as to maximize network utility, which is defined as the sum of all users' utilities. The utility function can be interpreted as either satisfaction (or quality of service) of a user or a knob to control efficiency and fairness of resource allocation. In general, this framework is formulated as a nonlinear optimization problem, in which the convexity of the problem is a critical property to solve it. In most cases, if the utility function is represented by a concave function, the convexity conditions of the problem are satisfied, as in [1], [2]. However, the concavity condition of the utility function is too restrictive, since in many cases, non-concave utility functions are more appropriate to model the characteristics of services [3], [4], [5], especially those of wireless services [3], [4]. Hence, it is important to deal with non-concave utility functions appropriately in the network utility maximization problem. However, in this case, the optimization problem is formulated as a non-convex problem, which is, in general, difficult to solve.

Fortunately, in most cases, we do not have to deal with the utility function with a very complex shape. The most popular types of utility functions in network utility maximization are concave, convex, S-shaped, and inverse-S-shaped (IS-shaped) functions, as shown in Fig. 1. In [4], a downlink power allocation algorithm is developed considering only three types of utility functions, *i.e.*, concave, convex, and S-shaped utility functions. In [3], an uplink power allocation algorithm is developed considering only convex and IS-shaped utility functions.

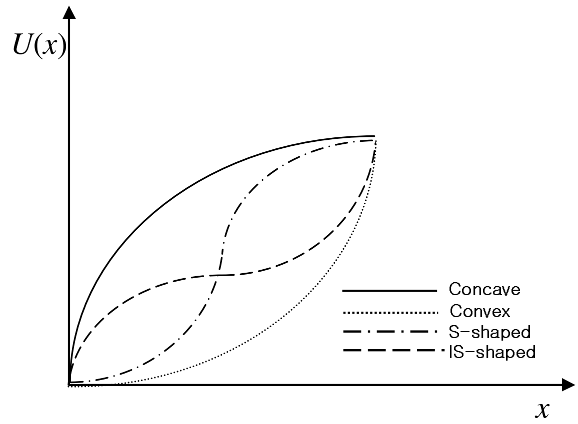

Fig. 1. Four types of utility functions

Hence, in [4] and [3], only a subset of four important types of utility functions are studied and their algorithms cannot be applied to other types of utility functions. For example, if we use the Shannon capacity as a utility function in the downlink power allocation problem, both algorithms in [4], [3] cannot be used, since it can be either a concave, convex, or IS-shaped function of the amount of power allocation depending on system parameters, even though it is a concave function of signal to interference and noise ratio (SINR) (see Fig. 4). Hence, in this paper, we study the power allocation problem considering all four types of utility functions. By utilizing special structures of those types of utility functions, we develop a simple and efficient algorithm that can deal with them together in a unified way.

The rest of the paper is organized as follows. In Section II, we first formulate the basic non-convex optimization problem that is studied in this paper and develop the algorithm to solve it. We show how this basic optimization problem is applied to the downlink power allocation problem in Section III. In Section IV, we provide numerical results and finally, we conclude in Section V.

II. Basic Optimization Problem and Its Solution

We first study a basic non-convex optimization problem that accommodates four types of utility functions (*i.e.*, concave, convex, S-shaped, and IS-shaped utility functions, as in Fig. 1) and develop a simple and efficient algorithm that solves it. The problem that we study in this paper is formulated as

$$\begin{array}{ll} \text{maximize}_{\mathbf{x}} & \sum_{i=1}^{N} U_i(x_i) \\ \text{subject to} & \sum_{i=1}^{N} x_i \le C, \\ & 0 \le x_i \le C, \ \forall i. \end{array} \quad (1)$$

In the above problem, we may think that U_i is the utility function of user i, x_i is the resource allocation for user i, $\mathbf{x}$ is a resource allocation vector, N is the number of users in the system, and C is the total amount of the resource in the system. We assume that the utility function U_i has the following properties.

(a) U_i is an increasing function of x_i.
(b) U_i is twice continuously differentiable.
(c) $U_i(C) < \infty$
(d) U_i is either convex, concave, S-shaped, or IS-shaped.

Since U_i can be non-concave, problem (1) is a non-convex optimization problem, which is, in general, difficult to obtain the optimal solution. Hence, in this paper, instead of developing a complex algorithm for the optimal solution, we will develop a simple algorithm that provides an efficient solution in the sense that network utility obtained from our algorithm is close to that obtained from the optimal solution. To this end, we will use a dual approach.

We first formulate the Lagrangian function and the dual objective function associated with problem (1), respectively, as

$$L(\mathbf{x}, \lambda) = \sum_{i=1}^{N} U_i(x_i) + \lambda(C - \sum_{i=1}^{N} x_i),$$

where λ is a Lagrange multiplier, and

$$Q(\lambda) = \max_{0 \le x_i \le C, \ \forall i} L(\mathbf{x}, \lambda). \quad (2)$$

Then, the dual problem is defined as

$$\min_{\lambda \ge 0} Q(\lambda). \quad (3)$$

This dual problem can be solved relatively easily by using standard methods such as the gradient projection and penalty algorithms (i.e., the dual optimal solution is relatively easily obtained). However, here, we are interested in the primal solution (i.e., resource allocation, ($\mathbf{x}$) rather than the dual solution (λ)). Especially, it is important to obtain an efficient and feasible primal solution that provides a high network utility and satisfies constraints in problem (1). However, due to duality gap and non-uniqueness of the primal solution obtained from the dual approach in the case of the non-convex optimization, simply solving the dual problem does not guarantee to obtain the efficient and feasible primal solution. To cope with this difficulty, we first consider problem (2) and study the properties of its solution

$$\mathbf{x}(\lambda) = \operatorname*{argmax}_{0 \le x_i \le C, \ \forall i} L(\mathbf{x}, \lambda).$$

Since the above problem is separable, each element $x_i(\lambda)$ of $\mathbf{x}(\lambda)$ is obtained by solving

$$x_i(\lambda) = \operatorname*{argmax}_{0 \le x_i \le C} \{U_i(x_i) - \lambda x_i\}, \ \forall i. \quad (4)$$

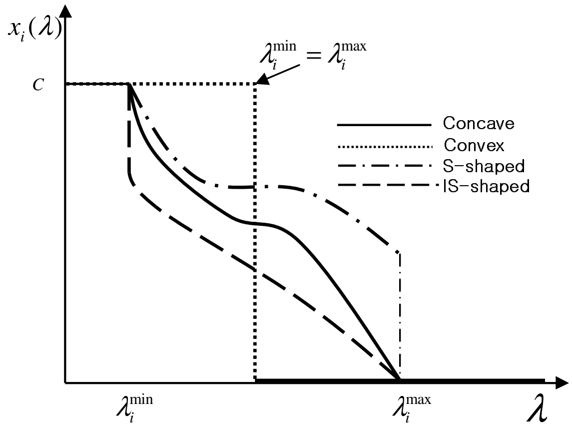

Fig. 2. The graph of $x_i(\lambda)$ for each type of utility functions

The properties of $x_i(\lambda)$ play a critical role to obtain the algorithm and they are well studied in [4], when U_i is either convex, concave, or S-shape. In this paper, since we also consider IS-shaped utility functions whose $x_i(\lambda)$ has different properties from those of the other shaped utility functions and has not been studied in [4], we also provide its properties. They can be studied in a similar way to those for the other types of utility functions, as in [4]. We first define λ_i^{max} and λ_i^{min} as

$$\lambda_i^{max} = \min\left\{\lambda \ge 0 \mid \max_{0 \le x_i \le C} \{U_i(x_i) - \lambda x_i\} = 0\right\}$$

and

$$\lambda_i^{min} = \max\{\lambda \ge 0 \mid x_i(\lambda) = C\}.$$

Then, the properties of $x_i(\lambda)$ is summarized as:

(a) $x_i(\lambda)$ is positive, continuous, and decreasing function of λ for $\lambda_i^{min} < \lambda < \lambda_i^{max}$.
(b) $x_i(\lambda) = 0$ for $\lambda > \lambda_i^{max}$ and $x_i(\lambda) = C$ for $\lambda < \lambda_i^{min}$.
(c) If U_i is concave, $x_i(\lambda)$ is a continuous function for all $\lambda \ge 0$
(d) If U_i is convex, $\lambda_i^{max} = \lambda_i^{min}$ and $x_i(\lambda_i^{max})(i.e., x_i(\lambda_i^{min}))$ has two values zero and C.
(e) If U_i is S-shaped, $x_i(\lambda)$ is discontinuous and has two values (zero and positive) at $\lambda = \lambda_i^{max}$.
(f) If U_i is IS-shaped, $x_i(\lambda)$ is discontinuous and has two values (positive and C) at $\lambda = \lambda_i^{min}$.

Fig. 2 shows the graph of $x_i(\lambda)$ for each type of utility functions.

Hence, $x_i(\lambda)$ can have two values depending on the value of λ and the shape of the utility function. In the following, we denote the solutions of (4) with $x_i^u(\lambda)$ and $x_i^l(\lambda)$, where $x_i^u(\lambda) > x_i^l(\lambda)$, if it has two solutions. If it has only one solution we denote it with $x_i(\lambda)$, $x_i^u(\lambda)$, or $x_i^l(\lambda)$, interchangeably and they represents the same value. Due to the discontinuity at λ_i^{min} in the case of the IS-shape utility function (property (f)), which does not happen in the other type of utility functions, we cannot directly apply the algorithm developed in [4] for our case and we will develop a new algorithm.

We now go back to dual problem (3). By the theory of the subdifferential [6], dual solution λ^* solves the dual problem if and only if

$$0 \in \partial Q(\lambda^*),$$

where $\partial Q(\lambda^*)$ is the subdifferential of $Q(\lambda)$ at λ^*, and it is obtained as

$$\partial Q(\lambda) = \left\{ d \mid C - \sum_{i=1}^{N} x_i^u(\lambda) \le d \le C - \sum_{i=1}^{N} x_i^l(\lambda) \right\}.$$

In this dual approach, after finding such a λ^*, we take $\mathbf{x}(\lambda^*)$ in (4) as the primal solution. However, since $x_i(\lambda^*)$ may not be unique as shown in the properties of $x_i(\lambda)$, some choices among them result in infeasible solutions (*i.e.*, $\sum x_i(\lambda^*) > C$) or smaller network utility than the other choices. Hence, in the following, we develop an algorithm that provides a feasible and efficient primal solution corresponding to the optimal dual solution. In fact, the main reason that causes the difficulty in solving this problem is the discontinuity of $x_i(\lambda)$ for the non-concave utility function. To deal with it, we first define λ_i^{dc} as

$$\lambda_i^{dc} = \begin{cases} \lambda_i^{max}, & \text{if } U_i \text{ is S-shaped} \\ \lambda_i^{max}(=\lambda_i^{min}), & \text{if } U_i \text{ is convex} \\ \lambda_i^{min}, & \text{if } U_i \text{ is inverse-S-shaped} \\ 0, & \text{if } U_i \text{ is concave} \end{cases}.$$

Hence, λ_i^{dc}'s are points at which $\sum x_i(\lambda)$ is discontinuous, including the boundary point (zero). In addition define $\lambda_0^{dc} = \infty$ and $\lambda_{N+1}^{dc} = 0$. Then, $x_i(\lambda_0^{dc}) = 0$ and $x_i(\lambda_{N+1}^{dc}) = C$. Without loss of generality, we assume that $\lambda_0^{dc} \ge \lambda_1^{dc} \ge \lambda_2^{dc} \ge \cdots \ge \lambda_N^{dc} \ge \lambda_{N+1}^{dc}$. We now develop the algorithm as follows.

Algorithm

(a) Let $k = 1$ and go to (b).

(b) If $\sum_{i=1}^N x_i^u(\lambda_k^{dc}) < C$, then go to (c).
If $\sum_{i=1}^N x_i^l(\lambda_k^{dc}) > C$, then go to (d).
If $\sum_{i=1}^N x_i^u(\lambda_k^{dc}) \ge C$ and $\sum_{i=1}^N x_i^l(\lambda_k^{dc}) \le C$, then go to (e).

(c) In this case, the dual optimal solution $\lambda^* < \lambda_k^{dc}$. Hence, let $k = k + 1$ and go to (b).

(d) In this case, since $\sum_{i=1}^N x_i^u(\lambda_{k-1}^{dc}) < C$ and $\sum_{i=1}^N x_i^l(\lambda_k^{dc}) > C$, we have $\lambda_{k-1}^{dc} < \lambda^* < \lambda_k^{dc}$. Since $x_i(\lambda)$'s are decreasing and continuous in this interval, we can find the dual optimal solution λ^* that satisfies $\sum_{i=1}^N x_i(\lambda^*) = C$ easily by using a standard method such as a bisection algorithm. Let $x_i^* = x_i(\lambda^*)$ for all i and the algorithm stops.

(e) In this case, the dual optimal solution $\lambda^* = \lambda_k^{dc}$ and the corresponding primal solution may not be unique. To find the efficient and feasible primal solution, First, let $x_i^* = x_i^u(\lambda^*)$, then obviously, $\sum_{i=1}^N x_i^* \ge C$.
- If $\sum_{i=1}^N x_i^* = C$, then the algorithm stops.
- Otherwise, select a user j randomly who has $x_j^* = x_j^u(\lambda^*)$ and $x_j^u(\lambda^*) \ne x_j^l(\lambda^*)$, and let $x_j^* = x_j^l(\lambda^*)$.
  - If $\sum_{i=1}^N x_i^* \le C$ then the algorithm stops.
  - Otherwise, repeat this procedure with another user while $\sum_{i=1}^N x_i^* > C$.

After the algorithm stops, we take $\mathbf{x}^*$ as the primal solution. Since we ensure that $\sum_{i=1}^N x_i^* \le C$ in the algorithm, it is a feasible solution. Moreover, if $\sum_{i=1}^N x_i^* = C$, then $\mathbf{x}^*$ is the optimal primal solution [4]. However, otherwise, it may not be the optimal solution. Hence, we now show the efficiency of $\mathbf{x}^*$ obtained in our algorithm. Let $\mathbf{x}^o$ be the optimal primal solution.

Theorem 1:

$$\sum_{i=1}^N U_i(x_i^o) - \sum_{i=1}^N U_i(x_i^*) < \max_i\{U_i(C)\}.$$

Proof: We only have to consider the situation when $\sum_{i=1}^k x_i^l(\lambda_k^{max}) < C$ and $\sum_{i=1}^k x_i^u(\lambda_k^{max}) > C$ in case (e), since in other cases, $\sum_{i=1}^N x_i^* = C$, which implies that $\mathbf{x}^*$ is the optimal solution.

In this case, due to the way we choose $\mathbf{x}^*$ in (e), there exists user k that has the following properties:

- $x_k^* = x_k^l(\lambda^*) < x_k^u(\lambda^*)$.
- $\sum_{i=1}^N x_i' > C$, where

$$x_i' = \begin{cases} x_i^*, & \text{if } i \ne k \\ x_i^u(\lambda^*), & \text{if } i = k \end{cases}.$$

Note that $\mathbf{x}'$ is still the primal solution corresponding to the dual optimal solution λ^*, but it is infeasible. Moreover,

$$\sum_{i=1}^N U_i(x_i') - \sum_{i=1}^N U_i(x_i^*) = U_k(x_k') - U_k(x_k^*). \quad (5)$$

By the weak duality theorem,

$$\sum_{i=1}^N U_i(x_i') + \lambda^*(C - \sum_{i=1}^N x_i') \ge \sum_{i=1}^N U_i(x_i^o).$$

Since $C - \sum_{i=1}^N x_i' < 0$ and $\lambda^* > 0$,

$$\sum_{i=1}^N U_i(x_i') > \sum_{i=1}^N U_i(x_i^o)$$

and from (5),

$$\begin{aligned} \sum_{i=1}^N U_i(x_i^o) - \sum_{i=1}^N U_i(x_i^*) &< U_k(x_k') - U_k(x_k^*) \\ &\le U_k(C) \\ &\le \max_i\{U_i(C)\}. \quad (6) \end{aligned}$$

■

Hence, the above theorem implies that the difference between the achieved utilities obtained by the optimal solution and our solution is at most the utility of one user. By using this property, we now have the following theorem, which implies that our solution is the asymptotically optimal solution.

Theorem 2: If $\sum_{i=1}^N U_i(x_i^o) \to \infty$ as $N \to \infty$, then

$$\frac{\sum_{i=1}^N U_i(x_i^*)}{\sum_{i=1}^N U_i(x_i^o)} \to 1, \text{ as } N \to \infty.$$

Proof: From (6),

$$\frac{\sum_{i=1}^N U_i(x_i^*)}{\sum_{i=1}^N U_i(x_i^o)} + \frac{U_k(x_k') - U_k(x_k^*)}{\sum_{i=1}^N U_i(x_i^o)} > 1$$

and

$$\frac{\sum_{i=1}^{N} U_i(x_i^*)}{\sum_{i=1}^{N} U_i(x_i^o)} > 1 - \epsilon,$$

where $\epsilon \to 0$ as $N \to \infty$, since $U_k(x_k') - U_k(x_k^*)$ is bounded and $\sum_{i=1}^{N} U_i(x_i^o) \to \infty$ as $N \to \infty$ by the assumption. Hence, by the fact that $\mathbf{x}^o$ is the optimal solution, the proof is completed. ∎

Remark 1: Even though the proposed algorithm already provides an efficient and feasible resource allocation $\mathbf{x}^*$, if $\sum_{i=1}^{N} x_i^* < C$, by allocating the remaining resource to users, we can further improve the system efficiency. There are several approaches for the redistribution of the remaining resource. But it is not an important issue in this paper and we do not discuss it.

III. Downlink Power Allocation

In this section, we study how the result from the previous section can be applied for the downlink power allocation problem. In this paper, we consider a single cell problem in which intercell interference is given and is not controllable.

In fact, in wireless systems, the utility (*i.e.*, performance) of a user depends on its SINR. Hence, in this section, we assume that the utility function of user i, U_i is a function of its SINR, γ_i. For the downlink case, the SINR of user i, γ_i is represented as

$$\gamma_i(\mathbf{p}) = \frac{n_i g_i p_i}{\theta g_i(\sum_{j=1}^{N} p_j - p_i) + I_i}, \tag{7}$$

where p_j is power allocation for user j, g_i is path gain between the base-station and user i, n_i is a constant (*e.g.*, processing gain) for user i, I_i is intercell interference and background noise for user i, and θ is orthogonality factor. Since the SINR is a function of the power allocation vector of all users in a cell, the utility function of a user is also a function of not only its own power allocation but also power allocation of the other users, which is a different form of the utility function in the previous section, in which the utility function is assumed to be a function of only its own resource allocation. However, as shown in [4], in the single cell downlink case, to maximize the network utility, the base-station must transmit at its maximum power level, *i.e.*, the summation of transmission powers of all users must be equal to the maximum transmission power of the base-station, p_T. Hence, we can rewrite the SINR of user i as

$$\gamma_i(p_i) = \frac{n_i g_i p_i}{\theta g_i(p_T - p_i) + I_i},$$

and the optimization problem for the downlink power allocation is written as

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{N} U_i'(p_i) \\ \text{subject to} & \sum_{i=1}^{N} p_i \le p_T, \\ & 0 \le p_i \le p_T, \ \forall i, \end{array}$$

where $U_i'(p_i) = U_i(\gamma_i(p_i))$. This problem can be solved by using the algorithm developed in the previous section, if $U_i'(p_i)$ is one of four types considered in the previous section, which can be applied for most cases [3], [4].

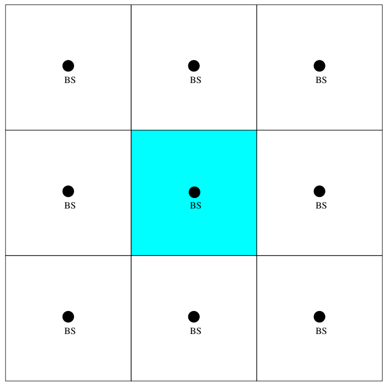

Fig. 3. The Cellular network.

IV. Numerical Results

In this section, we provide numerical results of our power allocation scheme for the downlink in the CDMA network. Hence, the parameters n_i and γ_i in (7) correspond to the processing gain and the bit energy to interference density ratio (E_b/I_0) for mobile i, respectively. For simplicity, we model the cellular system with nine square cells, as shown in Fig. 3. We assume that the base-station is located at the center of each cell and that each base-station has the same maximum power limit, p_T. We focus on the cell at the center of the system assuming that the base-stations in the other cells transmit at the maximum power level, p_T. We model the path gain from a base-station i to a mobile j, $G_{i,j}$ as follows:

$$G_{i,j} = \frac{K_{i,j}}{d_{i,j}^{\alpha}},$$

where $d_{i,j}$ is the distance from the base-station i to mobile j, α is a distance loss exponent, and $K_{i,j}$ is the log-normally distributed random variable with mean 0 and variance σ^2 (dB) that represents shadowing. For the simulation, we use a log utility function, which represents the Shannon's capacity and is expressed as

$$U_i(\gamma_i) = \log(1 + \gamma_i). \tag{8}$$

Hence, the utility function as a function of power allocation is represented as

$$\begin{aligned} U_i'(p_i) &= \log(1 + \gamma_i(p_i)) \\ &= \log\left(1 + \frac{n_i g_i p_i}{\theta g_i(p_T - p_i) + I_i}\right). \end{aligned} \tag{9}$$

One interesting fact of the utility function in (9) is that its shape can be either concave, convex, or IS-shaped depending on parameters in (9) such as the spreading gain, the orthogonality factor, and the channel condition, even though the function in (8) is a concave function of γ_i. We provide the examples for the shape of the utility function in Fig. 4 varying n_i and I_i/g_i, where n_i corresponds to the spreading gain and I_i/g_i corresponds to the channel condition. Moreover, since the channel condition of a user is time-varying, even though other parameters are fixed, the shape of the utility function of a user is also time-varying depending on its channel condition.

In Fig. 5, we show the performance of our algorithm. To show the efficiency of the proposed algorithm, we

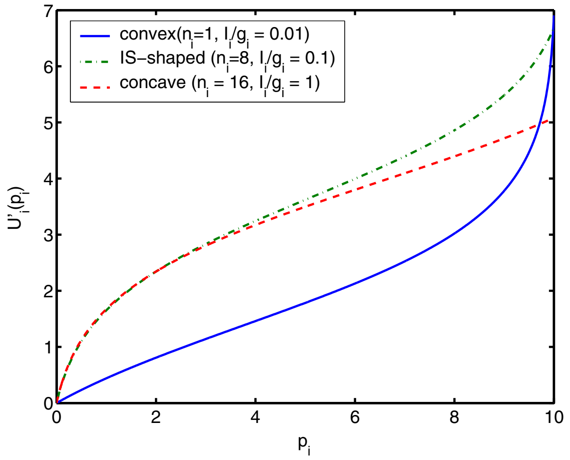

Fig. 4. The shape of the Shannon capacity as a function of power allocation ($\theta = 0.3$, and $p_T = 10$).

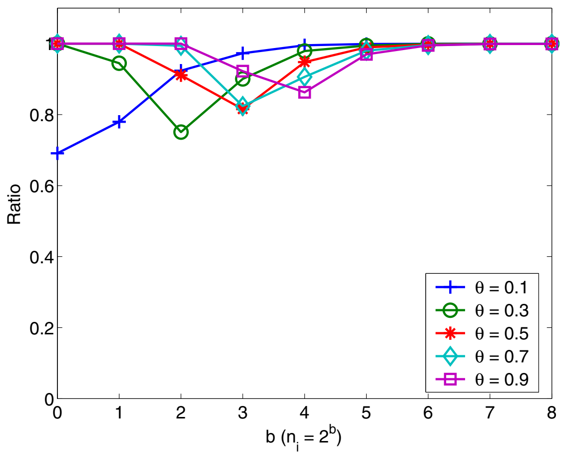

Fig. 5. The ratio of the total utility obtained by the proposed algorithm to the upper bound.

compare its performance with that of the upper bound on the optimal power allocation. The upper bound is obtained by solving a convex optimization problem in which each original non-concave utility function $U(p)$ is replaced with a concave function $V(p)$ that dominates the original non-concave utility function for $0 \le p \le P_T$, *i.e.*,

$$V(p) \ge U(p),\ 0 \le p \le P_T.$$

We set $P_T = 10$, $\alpha = 4$, $\sigma^2 = 8$. We also set the length of the side of the cell to be 1000 and the number of users in the cell to be 20. The figure shows the ratio of the achieved total utility obtained by the proposed algorithm to the upper bound varying θ, which corresponds to the orthogonality factor and $n_i = 2^b$, which corresponds to the spreading factor. For each experiment, we run the simulation program 10^3 times and plot the average values (At each time epoch of the simulation, each mobile is generated at a new location (with new path gain) in the cell via an independent uniform distribution. As the figure shows, in most cases, our algorithm provides the performance that is very close to the upper bound. Especially, when b is large (*i.e.*, when n_i is large) , our algorithm provides almost the same performance to the upper bound. [1]

V. Conclusion

In this paper, we studied power allocation problem by using network utility maximization framework. Compared with previous works that considered only limited types of utility functions, in this paper, we developed a simple and efficient power allocation algorithm considering four types of utility functions with which most general cases can be covered. Even though we take the power allocation problem for the application of our algorithm, our framework has been developed from a general optimization problem. Hence, our algorithm can be applied to other resource allocation problems, if their optimization problem has the same structure as that of the problem studied in this paper.

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[1] Since this figure shows the performance difference between the solution from our algorithm and the upper bound on the optimal solution, the performance difference between the solution from our algorithm and the optimal solution will be smaller than the difference shown in this figure.