Finite Support 2-D FIR QMF PR Filter Bank Design By Spectral Factorization Using Gröbner Basis

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***Abstract*—In this paper, the application of spectral factorization by using Gröbner basis approach to the problem of 2-D finite support QMF filter bank design is proposed along with illustrative example. Earlier, it has been shown that the Gröbner basis can be applied to solve the problem of 2-D factorization and its effectiveness in dealing with polynomial equation system are demonstrated. Despite the fact that there exist many good techniques for designing one dimensional QMF synthesis and analysis filter banks, the two-dimensional counterpart has a major bottleneck due to the generalization of 2-D spectral factorization. The proposed algorithm is the direct generalization of the 1-D QMF filter bank design by spectral factorization. Furthermore, the filter coefficients can be computed in symbolic forms and thus provide a good opportunity for further optimization if additional criteria are given.**

I. Introduction

Filter bank plays important roles in signal decomposition, signal compression, and signal analysis. The problem of QMF analysis and synthesis filter bank design has long been considered by various researchers, namely, Barnwell, Basu, Bose, Croisier, Daubechies, Esteban, Haar, Johnson, Kovacevic, Mitra, Mitzner, Smith, Strang, Vaidyanathan, Vetterli, Yamada, and more. In subband coding, the discrete-time signal to be encoded are divided into subband signals by using analysis filter bank. Before transmission, these subband signals are encoded, possibly with different allotted bit budgets. At the receiver, the received subband signals are reconstructed and combined to form the reconstructed signal by using synthesis filter bank. The performance of the subband coding scheme largely depends on the characteristics of analysis and synthesis filter banks. Some of the desired characteristics of practical filter banks include being distortionless, being aliasing-free, having linear-phase response, yielding perfect reconstruction (PR), and having a sharp cutoff transition bands.

In this paper, the 2-D QMF filter bank design procedure based on spectral factorization technique using Gröbner basis is proposed. The proposed design procedure can be generalized in to multidimensional case and possibly into the multiband case (with substantial modification). In this paper, the optimality is not a major concern yet and will be considered in the later occasion. In the next section, the fundamental concept of 2-D QMF filter bank and its desired properties are delineated. Illustratively, the design algorithm is then presented along with the example in Section III - V. Finally, the conclusion is given in Section V.

II. Two-dimensional QMF Filter Bank

In two-dimensional filter bank design, a specific kind of sampling scheme must be first chosen. To prevent loss of information, the number of subbands in a filter bank should be equal to the absolute of the determinant of sampling matrix S [5].

For a simple case, a *"quincunx"* sampling scheme with sampling matrix S, given by

$$S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{1}$$

is considered. Figure.1 shows the typical structure of 2-D

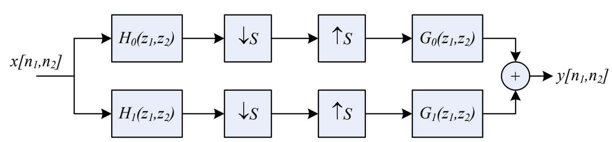

Fig. 1. The 2-D two-channel QMF bank structure.

QMF filter bank with quincunx sampling. As in 1-D case, it follows that the reconstructed output, in time domain, $y[n_1, n_2]$, can be expressed in z-transform domain as

$$\begin{aligned} Y(z_1,z_2) &= \frac{1}{2}\begin{bmatrix} G_0(z_1,z_2) & G_1(z_1,z_2) \end{bmatrix} \\ &\times \begin{bmatrix} H_0(z_1,z_2) & H_0(-z_1,-z_2) \\ H_1(z_1,z_2) & H_1(-z_1,-z_2) \end{bmatrix} \\ &\times \begin{bmatrix} X(z_1,z_2) \\ X(-z_1,-z_2) \end{bmatrix}. \end{aligned} \tag{2}$$

Typically, a two-band filter bank is required to satisfy a perfect reconstruction property, or equivalently, it must satisfy both the aliasing cancellation condition and the distortionless condition. Here, the review of the conditions are explained:

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A. Aliasing Cancellation Condition

The aliasing cancellation condition can be expressed as

$$H_0(-z_1,-z_2)G_0(z_1,z_2)+H_1(-z_1,-z_2)G_1(z_1,z_2)=0, \quad (3)$$

where

$$G_0(z_1,z_2) = H_1(-z_1,-z_2), \quad (4)$$
$$G_1(z_1,z_2) = -H_0(-z_1,-z_2). \quad (5)$$

The aliasing-free condition can be satisfied simply by choosing the synthesis filters G_0, G_1 to be antisymmetric/symmetric with the analysis filters, H_0, H_1 as in Eqs.(4)-(5)

B. Perfect Reconstruction Condition

If an alias-free QMF bank has no amplitude and phase distortions, then it is called a *perfect reconstruction* (PR) QMF bank and can be represented as

$$H_0(z_1,z_2)G_0(z_1,z_2)+H_1(z_1,z_2)G_1(z_1,z_2)=2z_1^{-l_1}z_2^{-l_2}. \quad (6)$$

where l_1 and l_2 are integers and cannot be negative, resulting in

$$Y(z_1,z_2)=z_1^{-l_1}z_2^{-l_2}X(z_1,z_2), \quad (7)$$

which in the time domain is equivalent to

$$y[n_1,n_2]=x[n_1-l_1,n_2-l_2] \quad (8)$$

for all possible inputs, indicating that the reconstructed output $y[n_1,n_2]$ is a delayed replica of the input $x[n_1-l_1,n_2-l_2]$.

The 2-D highpass analysis filter $H_1(z_1,z_2)$ can be represented as [11]

$$H_1(z_1,z_2)=-z_1^{-1}H_0(-z_1^{-1},-z_2^{-1}). \quad (9)$$

Then the perfect reconstruction condition in 2-D case is given by [3], [11]

$$P(z_1,z_2)+P(-z_1,-z_2)=2, \quad (10)$$

As represented in 1-D case [5], the transfer function of the 2-D half-band filter is adopt as

$$P(z_1,z_2)=H_0(z_1,z_2)H_0(z_1^{-1},z_2^{-1}). \quad (11)$$

In Eq.(11), given the appropriate 2-D polynomial $P(z_1,z_2)$, one may obtain the lowpass analysis filter by means of spectral factorization.

III. Main Result

The 2-D QMF PR filter bank design procedure is now outlined:

- **Step 1:** Design an appropriate half-band 2-D polynomial $P(z_1,z_2)$, such that its has 1) allowable support and 2) a lowpass response with the real positive definiteness property that is

$$0<P(z_1,z_2)<\infty, \quad \text{for,} \quad |z_1|=|z_2|=1.$$

- **Step 2:** Use the 2-D spectral factorization algorithm (based on the usage of Gröbner basis) to find all possible solutions of H_0.
- **Step 3:** Choose the best solution by considering its wave number response. If the additional performance constraints e.g., wave number response deviation, magnitude smoothness, are given, the selection can be done based on these constraints.

The detail derivation and design example are shown next.

Since the spectral factorization using Gröbner basis in one-dimensional case has already done for years [5], then the scope of work in this paper will be concentrated on two-dimensional case. We are trying to factor in the form according to Eq.(11) and must be satisfied the perfect reconstruction condition, i.e., Eq.(10). It was found that there are two main conditions that allow $P(z_1,z_2)$ to be factored which are 1) $P(z_1,z_2)$ is a 2-D zero-phase filter, and 2) $P(z_1,z_2)$ is positive on the unit bi-disc, $|z_1|=|z_2|=1$.

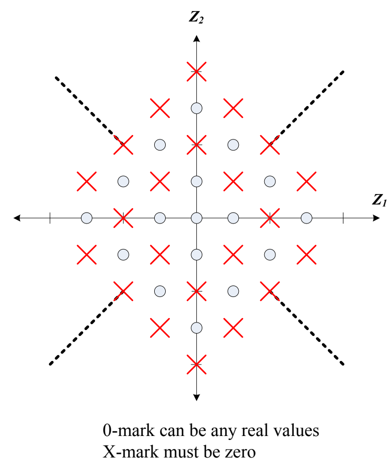

Fig. 2. The allowable support of $P(z_1,z_2)$ satisfied PR condition.

The support of $P(z_1,z_2)$ can be shown in Fig.2. It is clear from Eq.(10) that to satisfy the PR condition, not all positions in the support are allowed but only the ones that *total orders* of z_1 and z_2 are odd integers, i.e., the summation between the order of z_1 and z_2. Otherwise, the terms which contain the total orders in even number will be occurred and they cannot be cancelled to be satisfy the PR condition.

This is the hardest point because it is not simple as in 1-D case. For example, in 1-D case [5], we can assume that $H_0(z)=a+bz^{-1}+cz^{-2}+\ldots+kz^{-N}$ for $P(z)$ which has $(2N-1)$-tap size where $a,b,\ldots,k$ are *real constant* coefficients. When we find $P(z)$ and solve for its coefficients using Gröbner basis, the output system is a triangular form and we can easily find the results.

But in 2-D case, it is not that simple. We have to design $H_0(z_1,z_2)$ which makes $P(z_1,z_2)$ following to Fig.2. Therefore, we assume $H_0(z_1,z_2)$ to be

$$H_0(z_1,z_2)=az_1^{-1}z_2+bz_2+cz_1z_2+dz_1^{-1}+e+fz_1 +gz_1^{-1}z_2^{-1}+hz_2^{-1}+jz_1z_2^{-1}, \quad (12)$$

where $a,b,\ldots,j$ are any real numbers. For convenient, we write Eq.(12) in the coefficient matrix form following Fig.3

$$\mathbf{H_0}=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}, \quad (13)$$

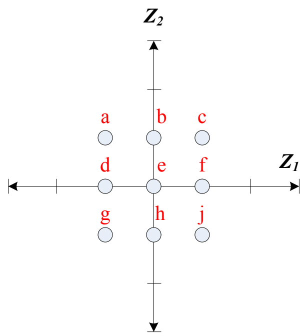

Fig. 3. The support of $H_0(z_1, z_2)$.

where row and column represent position of z_1 and z_2, respectively.

Next, we find $P(z_1, z_2)$ by multiplying $H_0(z_1, z_2)$ with $H_0(z_1^{-1}, z_2^{-1})$. Then the $P(z_1, z_2)$ in the form of matrix can be expressed as

$$\mathbf{P} = \begin{bmatrix} p_{(-2,2)} & p_{(-1,2)} & p_{(0,2)} & p_{(1,2)} & p_{(2,2)} \\ p_{(-2,1)} & p_{(-1,1)} & p_{(0,1)} & p_{(1,1)} & p_{(2,1)} \\ p_{(-2,0)} & p_{(-1,0)} & p_{(0,0)} & p_{(1,0)} & p_{(2,0)} \\ p_{(-2,-1)} & p_{(-1,-1)} & p_{(0,-1)} & p_{(1,-1)} & p_{(2,-1)} \\ p_{(-2,-2)} & p_{(-1,-2)} & p_{(0,-2)} & p_{(1,-1)} & p_{(2,-2)} \end{bmatrix},$$

where

$$\begin{aligned} & p_{(-2,2)} = p_{(2,-2)} = aj, \\ & p_{(-1,2)} = p_{(1,-2)} = bj + ah, \\ & p_{(0,2)} = p_{(0,-2)} = bh + cj + ag, \\ & p_{(1,2)} = p_{(-1,-2)} = hc + gb, \\ & p_{(2,2)} = p_{(-2,-2)} = gc, \\ & p_{(-1,1)} = p_{(1,-1)} = dh + ej + ae + bf, \\ & p_{(0,1)} = p_{(0,-1)} = eh + dg + be + fj + ad + cf, \\ & p_{(1,1)} = p_{(-1,-1)} = ec + db + hf + ge, \\ & p_{(2,1)} = p_{(-2,-1)} = dc + gf, \\ & p_{(0,0)} = a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^+h^2 + j^2, \\ & p_{(1,0)} = p_{(-1,0)} = de + ef + gh + hj + bc + ab, \\ & p_{(2,0)} = p_{(-2,0)} = df + gj + ac, \\ & p_{(2,-1)} = p_{(1,-2)} = dj + af, \end{aligned} \tag{14}$$

are *real* coefficients of matrix **P** which can be any numbers except the terms that total orders are even numbers, as follows

$$p_{(-2,2)} = p_{(0,2)} = p_{(2,2)} = 0, \tag{15}$$
$$p_{(-1,1)} = p_{(1,1)} = p_{(2,0)} = 0. \tag{16}$$

Then we use *"Singular Package"* [6] to compute the Gröbner basis of the rational ideal generated by the polynomials in system (15)-(16) with respect to the *lex ordering*.

The resulting Gröbner basis G consists of about 40 ideal generators but the problem is that the output system from Gröbner basis is not a triangular system anymore, so we have to add any conditions by trial and error to make system as a triangular system. For example, we add the condition in system (17) to system (15)-(16). Assumed that

$$a = b = c = d = j = 0. \tag{17}$$

Then the new system is a triangular form and it now can be solved. So, the output of this system is

$$ef + gh = 0. \tag{18}$$

which means that we are allowed to design totally 4 variables which are e, f, g, h and they can be any numbers that satisfy Eq.(18). The support of the output system is shown in Fig.4. While the support of the output $P(z_1, z_2)$ can be shown in Fig.5.

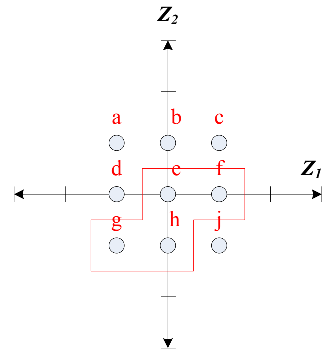

Fig. 4. The support of $H_0(z_1, z_2)$ which can be factorized.

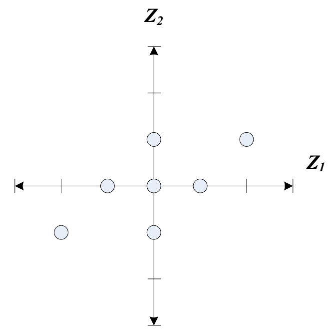

Fig. 5. The support of $P(z_1, z_2)$.

IV. Numerical Design Example of 2-D PR QMF Filter Bank

Given $P(z_1, z_2) = \frac{1}{196}(-3z_1^2 z_2 + 27z_2 + 24z_1^{-1} + 100 + 24z_1 + 27z_2^{-1} - 3z_1^{-2} z_2^{-1})$

Step 1: Define the coefficient matrix $\mathbf{H_0}$ as represented in Eq.(13).
Step 2: Find matrix **P** by multiplying matrix of $H_0(z_1, z_2)$ with matrix of $H_0(z_1^{-1}, z_2^{-1})$.
Step 3: Solving polynomial equations referring to Eq.(14) by using Gröbner basis, then

$$\begin{aligned} aj &= 0, \\ bj + ah &= 0, \\ bh + cj + ag &= 0, \\ hc + gb &= 0, \\ gc &= 0, \\ dh + ej + ae + bf &= 0, \\ eh + dg + be + fj + ad + cf &= 27/196, \end{aligned}$$

$$
\begin{aligned}
ec + db + hf + ge &= 0, \\
dc + gf &= -3/196, \\
a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^+ h^2 + j^2 &= 100/196, \\
de + ef + gh + hj + bc + ab &= 24/196, \\
df + gj + ac &= 0, \\
dj + af &= 0.
\end{aligned}
$$

The outputs, i.e., e, f, g, h, can be shown in the Table I

TABLE I
ALL POSSIBLE VALUES FOR e, f, g, h

Solution	e	f	g	h
1	0.2143	-0.0714	0.2143	0.6429
2	-0.2143	0.0714	-0.2143	-0.6429
3	0.6429	0.2143	-0.0714	0.2143
4	-0.6429	-0.2143	0.0714	-0.2143

Step 4: Choose the solution, in this experiment, solution 1 is chosen. Therefore, the matrix $\mathbf{H_0}$ can be written as

$$
\mathbf{H_0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.2143 & -0.0714 \\ 0.2143 & 0.6429 & 0 \end{bmatrix}, \qquad (19)
$$

or can be re-written in polynomial form as

$$
\begin{aligned}
H_0(z_1, z_2) &= 0.2143 - 0.0714 z_1 \\
&\quad + 0.2143 z_1^{-1} z_2^{-1} + 0.6429 z_2^{-1}. \qquad (20)
\end{aligned}
$$

An analysis highpass filter, i.e., Eq.(9), can be expressed as

$$
\begin{aligned}
H_1(z_1, z_2) &= -0.2143 z_1^{-1} - 0.0714 z_1^{-2} \\
&\quad - 0.2143 z_2 + 0.6429 z_1^{-1} z_2. \qquad (21)
\end{aligned}
$$

Then a synthesis lowpass filter G_0 in Eq.(4) can be represented as

$$
\begin{aligned}
G_0(z_1, z_2) &= 0.2143 z_1^{-1} - 0.0714 z_1^{-2} \\
&\quad + 0.2143 z_2 + 0.6429 z_1^{-1} z_2, \qquad (22)
\end{aligned}
$$

and a synthesis highpass filter G_1 from Eq.(5) can be formed as

$$
\begin{aligned}
G_1(z_1, z_2) &= -0.2143 - 0.0714 z_1 \\
&\quad - 0.2143 z_1^{-1} z_2^{-1} + 0.6429 z_2^{-1}. \qquad (23)
\end{aligned}
$$

The magnitude response of $H_0(z_1, z_2)$ and $H_1(z_1, z_2)$ can be shown in Fig.6 and Fig.7, respectively.

V. CONCLUSION

An optimal 2-D QMF PR analysis and synthesis filter bank can be designed based on the usage of 2-D factorization. With careful selection, an appropriate half-band polynomial $P(z_1, z_2)$ can be spectrally factorized into $H_0(z_1, z_2)$ and $H_0(z_1^{-1}, z_2^{-1})$ by using Gröbner basis. The solution set from the Gröbner basis algorithm is generalized and complete and can be modified to get the symbolic result, which allows further optimization for addition constraints. The proposed method has another advantage in that it can be generalized to design multi-dimensional filter banks.

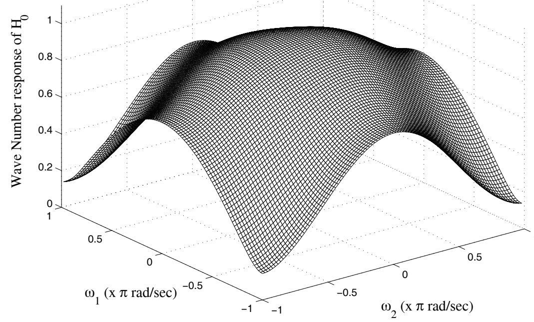

Fig. 6. Magnitude Response of a synthesis lowpass filter $H_0(z_1, z_2)$

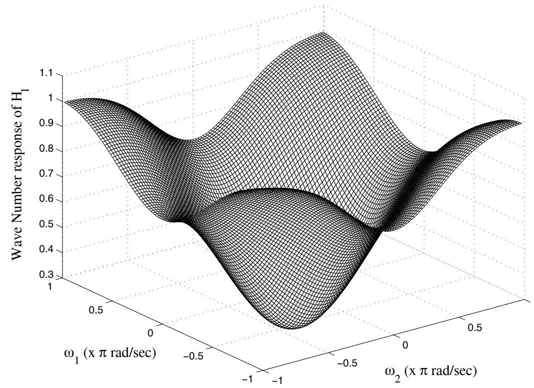

Fig. 7. Magnitude Response of a synthesis highpass filter $H_1(z_1, z_2)$

APPENDIX A
Introduction of Gröbner basis

The Gröbner basis is a particular kind of generating subset of an *ideal* I in a polynomial ring. There are too many applications of Gröbner basis in engineering field and one of the most applications is its ability to solve the system of polynomial equations.

Assumed that the *ideal* I is generated from a set of polynomials $f_1, f_2, \ldots, f_s$ which can be defined as

$$
I = \langle f_1, f_2, \ldots, f_s \rangle. \qquad (A1)
$$

The *variety* of the *ideal* I, $V(I)$, is the set of the solution of finite polynomial equation system

$$
f = 0, \quad \forall f \in I. \qquad (A2)
$$

Gröbner basis of an ideal I is a standard representation of that ideal by another equivalent ideal G,

$$
G = \langle g_1, g_2, \ldots, g_t \rangle. \qquad (A3)
$$

Since $G = I$, then $V(G) = V(I)$ and the solution of the new system is

$$
g = 0, \quad \forall g \in G. \qquad (A4)
$$

The lexicographical ordering is mostly used to solve the system of polynomial equations, then the resulting Gröbner basis will give in a form of a *triangular form* which can be easily solved as described in [5]. More details of Gröbner basis computation can be studied from [1] and [2].

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