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Fast Generation of Orthogonal Periodic Polynomials and Its Application to Smooth Approximation of the Internet Traffic

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***Abstract*— In this paper we propose a method to generate a sequence of periodic orthogonal polynomials over $[0,1]$, where polynomial weighting function is defined over the interval. Low computational load and numerical stability are realized by using the Lanczos's *three-term recurrence relation* to generate the sequence. Further improvement is provided by omitting numerical integration, and therefore, the proposed method requires only a small number of arithmetic operations. Finally the generated sequence is employed for least-squares smooth approximation of given traffic data having diel periodicity.**

I. Introduction

There are several periodicities in the Internet traffic because the users are dominated by common cycles such as the diel periodicity. For efficient adaptive network control in traffic engineering, one of the most important requirements would be estimation of such a trend in the traffic. The estimation can be essentially realized by approximation to the traffic with a low-frequency component having some specified periodicities. The smoothness can be characterized by several measurements such as support in the Fourier domain. Unfortunately, imposing periodicity is not straightforward if we employ such a transform using given fixed bases in transformed domain. Thus we need a method to design bases of the set of all periodic smooth functions, where such a method directly provides least-squares approximation of given function efficiently.

In this paper, we introduce a periodic polynomial approximation of given function, defined and periodic over closed interval, where smoothness is realized by restricting the degree of approximating polynomial. In this case, orthogonal bases are straightforwardly derived by applying the well-known Gram-Schmidt's orthogonalization procedure [1] to a sequence of periodic monomials with increasing degree. We show that it is possible to realize this approximation scheme computationally efficient and numerically stable. Firstly, we derive significant reduction of computational load and further numerical accuracy in generating the bases by introducing a *three-term recurrence relation* [1–3]. Indeed, this relation is employed in several areas, for example stable design of digital filters[4]. Secondly, we provide a simple equation that enables us to exactly compute each base with small number of arithmetic operations. Numerical experiment shows that smooth periodic approximation is derived for real traffic data.

II. Preliminaries

Let the set of all real numbers and integers be $\mathbb{R}$ and $\mathbb{Z}$, respectively. For all functions $f,g:[0,1]\to\mathbb{R}$, define their inner product by

$$\langle f,g\rangle := \int_0^1 w(x)f(x)g(x)\ \ x, \tag{1}$$

where $\int\cdot\ \ x$ means Lebesgue integral and $0<w(x)<\infty$ over $[0,1]$ is a weighting function. Define the induced norm by $\|f\|^2 := \langle f,f\rangle$ for all bounded function $f:[0,1]\to\mathbb{R}$. Then the set of all functions having bounded norm forms a Hilbert space. In this paper, we specially assume that $w(x)$ is a polynomial $\sum_{n=0}^{W} w_n x^n$ such that $w(x)>0$ for all $x\in[0,1]$. The set of all polynomials p such that $p(0)=p(1)=0$ and whose degrees are equal or less than N is defined by

$$\begin{aligned} &P_N \\ := &\left\{ \sum_{n=2}^{N} a_n x^{n-1}(x\quad 1) =: \sum_{n=2}^{N} a_n u_n(x) \;\middle|\; (a_n)_n \subset \mathbb{R} \right\} \\ = &\ \ \mathrm{span}\{u_n(x)\}_{n=2}^{N} \end{aligned}$$

The set of polynomials $(u_n)_{n=2}^N$ is linearly independent each other since an uniformly zero polynomial must be the constant zero itself. The following simple equation will be often used.

$$\int_0^1 x^k(x\quad 1)^2\ \ x = \frac{2}{(k+1)(k+2)(k+3)}$$

The network traffic has certain periodicity for example diel periodicity. Here we assume that, for given traffic data $\mathrm{trf}:[0,1]\to\mathbb{R}$, there exists expected value $\mathrm{e}_{01}\in\mathbb{R}$ around the edge of the interval. The problem to be finally resolved is:

$$\begin{aligned} &\text{find } p\in P_N \\ &\text{such that } \|(\mathrm{trf}\quad \mathrm{e}_{01})\quad p\| = \inf_{q\in P_N}\|(\mathrm{trf}\quad \mathrm{e}_{01})\quad q\| \end{aligned} \tag{2}$$

It is well-known that orthogonal bases of the subspace P_N is useful to resolve this least-squares approximation problem [5]. Such bases $(v_n)_{n=2}^N$ can be easily derived through application of the Gram-Schmidt procedure [5] as follows: $v_2=u_2$ and

$$v_n := u_n \quad \sum_{k=2}^{n-1}\frac{\langle u_n,v_k\rangle}{\|v_k\|^2}v_k \qquad (n\geq 3) \tag{3}$$

However, the Gram-Schmidt procedure consists of iterative scheme that requires to compute inner products with all derived orthogonal components. This implies that computational load rapidly becomes heavy as the degree N of P_N increases and accumulation of numerical error due to complicated computation.

III. Smooth Approximation of Traffic based on Fast Generation of Orthogonal Periodic Polynomials by the Lanczos's Three-Term Recurrence Relation

Multiplication of the variable x is a linear operator satisfying $\langle xf, g\rangle = \langle f, xg\rangle$ for any f, g (i.e. self-adjoint operator). This property and the linear independence of $(u_n)_{n\geq 2}$ allow us to introduce the Lanczos's three-term recurrence relation[1–3]. With the relation, we have the orthogonal polynomials $(v_n)_{n\geq 2}$ through the following equations

$$v_{n+1}(x) = (x \qquad {}_n)v_n(x) \quad \beta_n v_{n-1}(x) \quad (n = 2, 3, \quad), \tag{4}$$

where $v_2 = u_2, v_1 \equiv 0$ and

$${}_n := \frac{\langle xv_n, v_n\rangle}{\|v_n\|^2},\ \beta_n := \frac{\|v_n\|^2}{\|v_{n-1}\|^2} \quad (n = 2, 3, \quad) \tag{5}$$

Therefore, the original orthogonalization procedure in (3) is now reduced to that in (4). The modified formulation is known to be numerically stable and employed for digital filter design [4]. In addition to this simplification, we can derive further reduction of computational load. Note that

$$wv_m u_n \in P_{W+m+n}, \quad xwv_m u_n \in P_{W+m+n+1}$$

implies that application of (1) to (5) gives ${}_n$ and β_n only with small number of arithmetic operations.

The proposed smooth approximation method is summarized as follows.

Algorithm 1: For given weighting function w and maximum degree N, derive periodic orthogonal polynomials $(v_n)_{n=2}^N$ through (4). The optimal periodic polynomial p of (2) is given by

$$p = \sum_{k=2}^{N} \frac{\langle \mathrm{trf} \quad \mathrm{e}_{01}, v_k\rangle}{\|v_k\|^2} v_k$$

Then $p + \mathrm{e}_{01}$ is a smooth approximation of the traffic trf.

IV. Numerical Experiment

We employ "Leipzig.I" in NLANR PMA Project[6] as the traffic data to be approximated. The original data is converted into a step function where each step corresponds to total amount of traffic in 5 minutes. The converted data is shown in Fig. 2 as a rapidly fluctuating curve. The traffic around zero o'clock e_{01} is estimated through ensemble average for 4hours. We also assume an uniform weighting $w \equiv 1$ and the maximum degree $N = 10$. The proposed method in Algorithm 1 generates a set of orthogonal polynomials (Fig. 1) and provides a smooth approximation of the original traffic (smooth curve in Fig. 2).

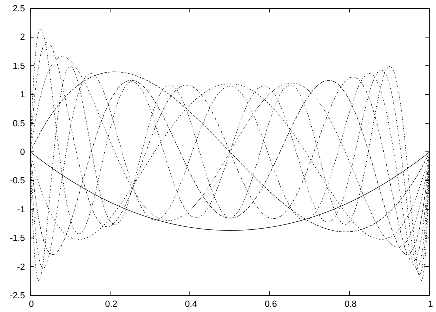

Fig. 1. Normalized Orthogonal Polynomials $\left(\frac{v_n}{\|v_n\|}\right)_{n=2}^{10}$

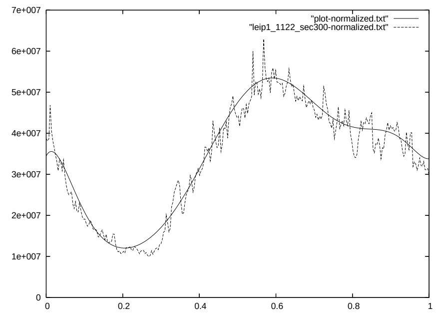

Fig. 2. Traffic and Its Smooth Approximation

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