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Generalised Weierstrass elliptic functions and nonlinear wave equations

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Abstract—The well-known Weierstrass elliptic functions are constructed from an algebraic curve of genus $g = 1$, and can be used to solve a number of nonlinear ordinary differential equations, such as the travelling wave problem for the KdV equation. As well as the soliton solution, such methods give periodic solutions of the ODEs. If the curve is generalised to a higher genus, the corresponding generalised Weierstrass functions give multiple periodic solutions of many well-known PDEs, such as the KdV equation ($g = 2$), the Boussinesq equation ($g = 3$), and the Kadomtsev-Petviashvili (KP) equation ($g = 6$). We review, very briefly, some of the results in this area.

1. Introduction

The well-known Weierstrass elliptic functions were first studied in the middle of the 19th century by Abel, Hermite and Weierstrass. The functions were related to an elliptic curve, given in standard Weierstrass form as

$$y^2 = 4x^3 - g_2 x - g_3, \tag{1}$$

where g_2 and g_3 are parameters. We refer to this as the (2,3) curve. If x and y are considered to be complex variables, the resulting surface has the shape of a torus, which has genus $g = 1$. Associated with the curve (1) is the Weierstrass $\wp(u)$ function, where the variable u satisfies $u = \int dx/y$, where dx/y is called a differential of the first kind. The doubly periodic function $\wp(u)$ satisfies the ODEs

$$\begin{aligned} (\wp')^2 &= 4\wp^3 - g_2\wp - g_3, & (2)\\ \wp'' &= 6\wp^2 - \frac{1}{2}g_2. & (3) \end{aligned}$$

The second ODE follows easily from the first after differentiation and cancellation. Additionally, we have the Weierstrass $\sigma(u)$ function, which is is related to $\wp$ by

$$\wp(u) = -\frac{\mathrm{d}^2}{\mathrm{d}u^2} \ln \sigma(u) \tag{4}$$

A well-known application of (2) is to the travelling wave solution of the Korteweg-de Vries (KdV) equation

$$u_t + 6u\, u_u + u_{xxx} = 0, \tag{5}$$

where the subscripts denote partial differentiation. Putting $u(x,t) = f(x - ct) = f(\zeta)$, we find

$$-c\frac{\mathrm{d}f}{\mathrm{d}\zeta} + 6f\frac{\mathrm{d}f}{\mathrm{d}\zeta} + \frac{\mathrm{d}^3 f}{\mathrm{d}\zeta^3} = 0. \tag{6}$$

Integrating with respect to ζ we get, on setting the constant of integration equal to zero

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\zeta^2} + 3f^2 - cf = 0, \tag{7}$$

which is just (3) with $f = -2\wp(u) + c/6, g_2 = c^2/12$.

We discuss below what happens when we generalise the curve (1) to higher genus curves, with powers of x and y greater than 2 or 3 respectively. We can then find generalisations of the ODEs (2) and (3) which are now PDEs in functions of g variables.

2. Genus 2 case

In genus 2 the relevant curve is hyperelliptic (leading term in y is y^2) (2,5) curve

$$C: \quad y^2 = x^5 + \mu_2 x^4 + \mu_4 x^3 + \ldots + \mu_0. \tag{8}$$

This case was considered in detail by Baker (1907). σ and $\wp$ are now functions of $g = 2$ variables, i.e.

$$\sigma = \sigma(u_1, u_2) = \sigma(\mathbf{u})$$

There are now two differentials of the first kind, dx/y and $x\,dx/y$, and we define u_1, u_2 by

$$\begin{aligned} u_1 &= \int^{(x_1,y_1)} \frac{dx}{y} + \int^{(x_2,y_2)} \frac{dx}{y}, & (9)\\ u_2 &= \int^{(x_1,y_1)} \frac{x\,dx}{y} + \int^{(x_2,y_2)} \frac{x\,dx}{y}, & (10) \end{aligned}$$

for two variable points (x_i, y_i) on C.

2.1. Genus 2 PDEs

Now that our function sigma depends on several variables, we need a new notation corresponding to the 2nd logarithmic derivative in (4). The generalized $\wp$ functions (note more than 1 type!) are defined from the σ function as

$$\wp_{ij}(u_1, u_2) \equiv -\frac{\partial^2}{\partial u_i \partial u_j} \ln \sigma(u_1, u_2), \quad i = 1, 2. \tag{11}$$

There is a nice if somewhat complicated expansion formula due to Klein to get the PDE's, once the curve and the differentials have been fixed. In the genus 2 case this gives

$$\begin{aligned}
\wp_{2222} - 6\wp_{22}^2 &= 2\mu_4 + 4\mu_2\wp_{22} + 4\wp_{12}, \\
\wp_{1222} - 6\wp_{22}\wp_{12} &= 4\mu_2\wp_{12} - 2\wp_{11}, \\
\wp_{1122} - 2\wp_{22}\wp_{11} - 4\wp_{12}^2 &= 2\mu_4\wp_{12}, \\
\wp_{1112} - 6\wp_{11}\wp_{12} &= 4\mu_2\wp_{12} - 2\mu_8\wp_{22} - 4\mu_{10}, \\
\wp_{1111} - 6\wp_{11}^2 &= \mu_8\wp_{12} + 4\mu_6\wp_{11} \\
&\quad -12\mu_{10}\wp_{22} - 2\mu_{10}\mu_2 - 2\mu_8\mu_4.
\end{aligned}$$

These are the generalization of $\wp'' - 6\wp^2 = -\frac{1}{2}g_2$ in genus one ($\wp_{1111} - 6\wp_{11}^2 = -\frac{1}{2}g_2$ in our new PDE notation).

If we take the first of these equations, differentiate once w.r.t. u_2, then put $u_1 = t, u_2 = x, \wp_{22} = -U(x,t), \mu_2 = 0$, we find

$$U_t + 12UU_x + U_{xxx} = 0,$$

which is the KdV equation. Our function $\wp_{22}$ is hence a fully two-dimensional multi-periodic solution of the KdV equation. This result was derived by Baker in 1907 [1], some 60 years before the discovery of the "modern" multi-soliton solutions of KdV.

The σ function in the genus 2 case can be expanded in a power series in u_1, u_2

$$\begin{aligned}
\sigma(u_1,u_2) &= u_1 - \frac{1}{3}u_2{}^3 - \frac{1}{60}\mu_2 u_2^5 \\
&\quad -\frac{1}{1260}(8\mu_2^2 + \mu_4)u_2^7 + \frac{1}{12}\mu_4 u_1 u_2^4 + \ldots
\end{aligned}$$

The function σ satisfies four *linear* heat-type equations in the u_i and the μ_j, the first of which is

$$\begin{aligned}
&4\mu_4\frac{\partial\sigma}{\partial\mu_4} + 6\mu_6\frac{\partial\sigma}{\partial\mu_6} + 8\mu_8\frac{\partial\sigma}{\partial\mu_8} + 10\mu_{10}\frac{\partial\sigma}{\partial\mu_{10}} \\
&= 3\,u_1\frac{\partial\sigma}{\partial u_1} + u_2\frac{\partial\sigma}{\partial u_2}\sigma - 3
\end{aligned}$$

These heat-type equations can be used to form recurrence relations satisfied by the coefficients of the sigma expansion.

Many results for the hyperelliptic cases $y^2 = x^s + \ldots$, including many general theorems, have been derived in [2].

3. Genus 3

Here we consider only the so-called *trigonal* genus 3 curve, which is non-hyperelliptic [3]

$$C: \quad y^3 = x^4 + \mu_3 x^3 + \ldots + \mu_0,$$

Now all functions are functions of $\mathbf{u} = (u_1, u_2, u_3)$, defined in an analogous way to (9). The resulting $\wp$ functions coming from this (3,4) curve satisfy a number of PDEs, the first of which is

$$\wp_{3333} - 6\,\wp_{33}^2 = -3\,\wp_{22},$$

which can be shown to transform to the Boussinesq equation

$$U_{tt} - U_{xx} - \frac{\partial}{\partial x^2}\left(3u^2 + U_{xx}\right) = 0$$

The corresponding σ function can again be shown to solve a set of heat-type equations, which can also be used to form a recurrence relations for the coefficients of the corresponding sigma expansion.

Similar results have been derived for higher genus trigonal curves $y^3 = x^5 + \ldots, y^3 = x^7 + \ldots$, etc.

4. Genus 6

In genus 6 a new possibility occurs, the so-called tetragonal (4,5) curve [4]

$$y^4 = x^5 + \ldots.$$

Many of the corresponding equations have been worked out in this case, although things get quite complicated. Now the $\wp$ and σ functions are functions of six variables $u_i, i = 1,\ldots,6$. The $\wp$ functions satisfy a number of PDEs, the first of which is

$$\wp_{6666} - 6\wp_{66}^2 = 3\wp_{55} + 4\wp_{46}. \qquad (12)$$

Differentiating (12) twice with respect to u_6 we get

$$\wp_{666666} = 12\frac{\partial}{\partial u_6}\wp_{66}\wp_{666} - 3\wp_{5566} + 4\wp_{4666}.$$

Making the substitutions $u_6 = x, u_5 = y, u_4 = t$, and $U(x,y,t) = \wp_{66}$, we get eventually the following parametrized form of the KP-equation:

$$(U_{xxx} - 12UU_x - 4U_t)_x + 3U_{yy} = 0.$$

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