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Space-Time analogy in delay systems for chimera states and Reservoir Computing

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Abstract—Space-Time analogy for nonlinear delay dynamics was proposed in the early 1990s, as a tool to visualize and interprete complex motions in the time domain only. Beyond the visualization tool, this analogy was recently pushed further as a conceptual argument to support the emulation, through nonlinear delay dynamics, of traditional spatio-temporal dynamics such as the ones provided by neural networks or by paradigmatic models like network of coupled Kuramoto oscillators. The contribution will report on experimental and analytical results obtained for both Reservoir Computing and chimera states, and showing the powerful capability of delay system to emulate features usually attributed to newtork of oscillators.

1. Introduction

In the field of complex systems, network of oscillators are representing a widely used paradigm in order to numerically explore the dynamical features issued from high dimensional systems. From a more applied perspective, such studies are expected to provide insights for the understanding of real worl complex systems, whether natural ones such as the brain, or technological ones such as power grid networks, among many other examples. For simplifications issues, partly motivated by more tractable problems addressed through numerical or theoretical approaches, perfectly identical oscillator networks have been also explored along the same research direction. Symmetry effects then are expected to come with more importance, however losing partly relevance with respect to realistic situation where small differences or variations are always experienced within a real world network of dynamical nodes.

We have recently contributed in two different research topics related to the oscillator network theory, through an unconventional virtual emulation of a network of oscillators by delay differential equations. Thanks to this virtual emulation, one has access to a network of rigorously identical oscillator, with stricly identical coupling. Moreover, due to the ease of physical implementation of delay systems, whether in electronic of photonics, such an approach could represent a very interesting new paradigm for the study oscillator networks. After a brief explanation of the theoretical concepts supporting the virtual emulation of oscillator networks through delay dynamics and thus the actual relevance of a space-time analogy for delay systems, we will report on the two topics through which we have tested this relevance : The study of chimera states occuring in identically coupled oscillators in a network, and the design and implementation of a novel neural network-based computing concept (Reservoir Computing).

2. Space-time analogy and impulse response modeling

We assume the system under study is belonging to the general class of Mackey-Glass or Ikeda delay dynamical systems [1, 2], where the dynamical variable is named $x(t)$, and the involved delay is τ_D. In such systems, one can conceptually split the feedback systems into two subsystems interacting circularly one with the other [3]: A linear dynamical part from which the dynamical variable $x(t) = \mathrm{FT}^{-1}[X(\omega)]$ is obtained (FT^{-1} stands for the inverse Fourier Transform), and which is modeled by a linear filtering $H(\omega)$ in the Fourier domain; And an adiabatic (i.e. instantaneous, without any dynamical effect) nonlinear delayed part represented by the feedback signal $z(t) = \mathrm{FT}^{-1}[Z(\omega)] = f_{\mathrm{NL}}[x(t-\tau_D)]$. With such an asumption, the modeling of the dynamics in the Fourier domain reduces to the very simple following equality:

$$X(\omega) = H(\omega) \cdot Z(\omega) \tag{1}$$

From the previous equation, and invoking the conversion rules from the Fourier domain to the time domain (e.g. $i\omega X(\omega) \to \mathrm{d}x/\mathrm{d}t$), one generally derives the delay differential equation ruling the dynamics of $x(t)$ solely (without $z(t)$ which can be now replaced by its definition depending on $x(t-\tau_D)$ only) in the time domain. Such a derivation is however subjected to the knowledge of the exact filtering profile for $H(\omega)$, typically in the form of a fraction of polynomials. The simplest form for such a filtering profile is the one of a low pass filter ($H(\omega) = (1 + i\omega\tau)^{-1}$), which results in the typical first order scalar delay differential equation as concerned with the Mackey-Glass or Ikeda models,

$$\tau \frac{\mathrm{d}x}{\mathrm{d}t}(t) = -x(t) + f_{\mathrm{NL}}[x(t-\tau_D)]. \tag{2}$$

Such models, despite their large interest and the many publications reporting on their complex dynamical behaviors, however represent a very specific sub-class only of the class of problems modeled in the Fourier domain by Eq.(1).

Each kind of linear Fourier filter thus leads to a new subclass of delay differential, as examplified by the bandpass filtering case our group has studied since the early 2000 [4], and which has revealed many new dynamical phenomena (chaotic breathers, Neymarc-Sacker bifurcation, single period periodic motion, chimera states) compared to the most widely studied low-pass case. The simplest bandpass type filter is $H(\omega) = i\omega\theta/[(1 + i\omega\theta)(1 + i\omega\tau)]$, where τ and θ are the characteristic time scales determining the high and low cut-off frequencies $f_h = (2\pi\tau)^{-1}$ and $f_l = (2\pi\theta)^{-1}$ respectively. Under this filtering model asumption, an integro-differential delay equation can be deduced:

$$\frac{1}{\theta}\int_{t_0}^{t} x(\xi)\,\mathrm{d}\xi + x(t) + \tau\,\frac{\mathrm{d}x}{\mathrm{d}t}(t) = z(t) = f_{\mathrm{NL}}[x(t-\tau_D)], \quad (3)$$

Beyond these many particular cases derived for each new Fourier filtering profile, one can keep in the time domain the generality offered by the Fourier domain through Eq.(1), however losing the convenient and widely preferred differential equation description for the time. The direct conversion of Eq.(1) indeed results in a convolution product (thus a “global” integral representation of the dynamics, instead of the local one provided by a differential equation):

$$x(t) = \int_{-\infty}^{t} h(t-\xi)\cdot f_{\mathrm{NL}}[x(\xi-\tau_D)]\,\mathrm{d}\xi, \quad (4)$$

where $h(t)$ is the well known (causal) impulse response of the linear filter defined as the inverse Fourier transform of the Fourier filtering function $H(\omega)$. Re-writing such an integral representation of the dynamics with some specific features known for delay equations, e.g. the actual infinite dimensionality of such dynamics because of the functional nature of its initial conditions (e.g. a function of time $x_0(t)$ defined over the time interval $t \in [-\tau_D, 0]$), one can obtain the following expression [5]:

$$x_n(\sigma) = x_{n-1}(\sigma) + \int_{\sigma-1}^{\sigma+\gamma} h(\sigma+\gamma-\xi)\cdot f[x_{n-1}(\xi)]\,\mathrm{d}\xi, \quad (5)$$

where the time is decomposed as follows, $t = (n\eta + \sigma)\tau_D$, with $n \in \mathbb{N}$ and with η being a constant close to unity, $\eta = 1 + \gamma$ (with $\gamma = \mathrm{o}(\tau/\tau_D)$), thus reflecting the time delay iteration process inherent to delay dynamics. From the previous equation which is rigorously derived analytically, one can clearly make a new physical interpretation in terms of network of coupled oscillators for the space-time analogy earlier proposed for delay equations [6]: The amplitude $x_n(\sigma)$ of any oscillator corresponding to a virtual position $\sigma \in [0, \eta]$, is dynamically ruled from the same amplitude $x_{n-1}(\sigma)$ at one time delay earlier (iteration from $(n-1)$ to n, resulting in a discrete time dynamics), with a modification ruled by the integral term. This integral term appears as a nonlinear coupling of the continuously distributed neighboring oscillators at positions ξ around σ, the impulse response h playing the role of a coupling coefficient.

In the next sections, we will illustrate for two particular situations, how such a space-time analogy was recently used on the one hand to discover the existence of chimera states in delay dynamics [5], and on the other hand to demonstrate the processing efficiency of delay systems when they are replacing the dynamics of a neural network to perform Reservoir Computing [7].

3. Chimera states in delay systems

Chimera states have been discovered numerically in 2002 by Kuramoto [8], while exploring the emergence of symmetry breaking solutions exhibited by network of coupled identical oscillators. In the case of long range (non local) coupling conditions, particular sustained solutions were observed, in which the whole network appears to be structured into sub-networks of congruent solutions within a sub-network, but incongruent between sub-networks. Chimera states have attracted lots of interest because of their non-intuitive features corresponding to symmetry breaking solutions within a network constructed with perfect symmetry. In the case of a network of phase oscillators, it can be found under appropriate coupling offset phase and coupling radius, that parts of the network exhibit fully synchronized oscillators whereas other parts show totally desynchronized ones. Both regions appear to coexist in a stable way within the whole network. After their first discovery, one had to wait 10 years until experimental observation of chimera states could be achieved in 2012. Two independent papers in two different fields, optics and chemistry, reported the experimental formation of such chimera patterns. One in a spatio-temporal dynamics of the intensity profile of a light beam, and another in the volume of a reactor where a Belousov-Zabotinsky chemical reaction was prepared.

In 2013 [9], based on the asumption that delay systems can mimick some features of spatio-temporal dynamics, we reported the first numerical and experimental observation of chimera state within the virtual space-time representation of this infinite dimensional dynamics. Such a representation precisely highlights the discrete time evolution along a vertical axis when n is incremented every time delay iteration, of the virtual spatial domain amplitude distribution within each time delay $\{x_n(\sigma) \mid \sigma \in [0, \eta]\}$. Under appropriate parameter condition and delay dynamical model, the corresponding functional $x(\sigma)$ evolving over time n was clearly exhibiting the emergence of a well structured virtual space along σ, with an alternance of quiet plateaux and chaotic-like oscillations, sustained within the “length” τ_D of the virtual space as the discrete time n is growing.

Figure 1 shows an example of an experimentally recorded scalar time trace $x(t)$ from a bandpass delay dynamical system. The time series was then cut according to poperly chosen “spatial” intervals such that each for $t \in [n\eta\tau_D; (n+1)\eta\tau_D]$, we stack vertically the color encoded amplitudes $x(t)$. From this representation, a particu-

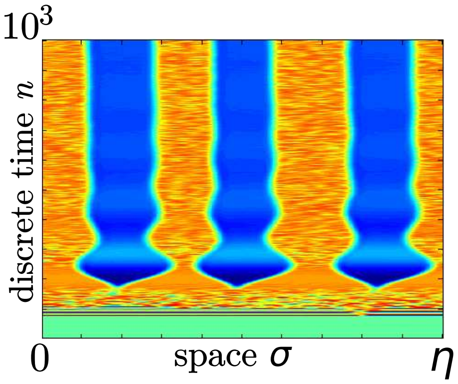

Figure 1: Space-Time plot of an experimental 3-headed chimera solution emerging (n growing from bottom to top) from a background noise.

lar pattern can be clearly viewed and identified as a chimera state.

Thanks to the derivation of Eq.(5), interesting analogies and interpretation have been proposed in terms of coupling distance and its influence on the chimera existence and feature [5]. Since the coupling function between distant virtual oscillators turns out to be determined by the impulse response profile $h(\xi)$, one can carve such a function directly through the Fourier filter used in the delayed feedback loop. Work is in progress to demonstrate, via the convolution product description involving the impulse response, why chimera state can be found (or are not stable) in low pass delay dynamics, whereas they have been indeed observed for bandpass delay dynamics. One can notice for example, that a direct consequence of a bandpass filter compared to a low pass, is to extend the equivalent coupling range in terms of network of oscillators, through a broader impulse response.

Chimera states reveals deterministic organization of complexity within high dimensional dynamical systems. They correspond to the emergence of spontaneous complex dynamics in an autonomous way, in the sense there is no information provided by the external world of the dynamics, except for the initial noisy background from which chimera appears. In the next section, similar spatio-temporal features of high complexity delay dynamics will be reported, however in a strongly non autonomous way. Indeed, we will report on the processing capability of delay dynamics while they are subject to large amplitude external forcing coming from the information signal to be processed.

4. Reservoir Computing (RC) with delay systems

The concept of RC [10, 11] is derived from recurrent neural network (RNN) approaches, however simplifying extremely the learning phase of the computational steps. The latter indeed represents traditionally a very critical issue in standard RNN, because the optimal set of coupling parameters is very difficult to determine by a learning procedure, particularly when they concern many sets of such connectivity strength, the ones of the input and output layers, and the ones of the internal connectivity defining the network structure itself. RC considers that the output connectivity, also called the read-out or output layer only, needs to be learnt. The two other sets of connectiviy coefficients do not need critical optimization, and they can be thus simply chosen at random for example. Such a simplification transforms the learning phase into a very simple, very efficient, very fast, and always converging solution. Beyond this surprising simplification, RC has moreover shown surprising computation accuracy, with comparable results, and sometimes even better ones, compared to traditional neural network computing.

More recently, RC has reached another important step forward through its successful hardware demonstration [12], moreover with an initially unexpected structural solution for the so-called Reservoir: the usual network of interconnected nodes was physically realized through the internal complexity of a delay dynamical system. As illustrated in the theoretical arguments of Section 2, delay dynamics can provide qualitatively similar complexity features compared to spatio-temporal dynamics such as a network of neurons. In the present section, we will again take the opportunity of the unusual modeling of delay dynamics through Eqs.(4) and (5), in order to analytically derive a rigorous correspondence between a delay dynamics seeded by a time division multiplexed input information, and a network of interconnected nodes excited by an input information through the usual input layer.

A key concept in the use of a delay system to emulate a neural network, is to consider the dynamical nodes of the network as being temporal positions within the time interval corresponding to the delay. One needs then to re-define the time variable t, so that it can reflect the emulation of a virtual spatial position $\sigma \in [0, \eta]$, which is updated in time each round trip of the signal in the delayed feedbak loop, i.e. each time delay τ_D. Such an approach indeed reveals the intrinsic multiple time scale feature of a delay dynamics, the fast time scale τ related to the high cut-off frequency f_h, and the slow time related to the delay τ_D: $t = (\sigma + n) \cdot \eta\tau_D$.

If one then assumes that the virtual nodes correspond to sampled positions $\sigma_k = k\,\delta\tau/\tau_D$, the number of virtual nodes in the delay dynamics amounts to $K = \tau_D/\delta\tau$. Addressing each of these nodes with an input vector $\mathbf{u}(n) \in \mathbb{R}^Q$ is achieved, as already stated, through a standard time division multiplexing technique. Distributing "randomly" each vector component of $\mathbf{u}(n)$ onto each of the K virtual nodes of the delay dynamics, is an operation typically performed according to a so-called input connectivity matrix. From Eq.(4), one can arrange the integration interval for the convolution product so that the node amplitude $x_k \equiv x_{\sigma_k}$ at time n can be expressed as an update of the amplitude of the same node, but at time $(n-1)$, i.e. $x_k(n-1)$. Taking also into account that the dynamics is seeded by the input information to be processed, one obtains:

$$x_k(n) = x_k(n-1) + \int_{\sigma_k - \tau_D}^{\sigma_k} h(\sigma - \sigma_k) \times \\ f_{\rm NL}\left[x_\sigma(n-1) + \rho \cdot u^I_\sigma(n-1)\right] \mathrm{d}\sigma. \qquad (6)$$

The latter expression reveals in a rigorous way the analogy of delay-based RC with the original Echo State Network approach as proposed in [10].
The output layer consists also in a matrix multiplication, corresponding physically to a circular convolution operated on the response signal $x_\sigma(n)$ and involving the Read-Out matrix $\mathbf{W}^{\rm R} = [w^{\rm R}_{mk}] \in \mathbb{R}^M \times \mathbb{R}^K$. The computed output is a vector $\mathbf{y}(n) \in \mathbb{R}^M$, which is the expected calculation result obtained from the input information $\mathbf{u}(n)$:

$$y_m(n) = \sum_{k=1}^{K} w^{\rm R}_{mk}\, x_k(n). \qquad (7)$$

The Read-Out matrix $\mathbf{W}^{\rm R}$ is practically the solution of a ridge regression problem minimizing the error of the output vector considering a set of known pairs of answer / response $(\mathbf{u}(n); \tilde{\mathbf{y}}(n))$. This ridge regression step precisely corresponds to the learning phase of such a delay-based RC.

This delay-based RC concept was practically implemented recently by different authors, with very successful computational performances. A classical speech recognition problem was for example performed experimentally [7, 12], with record word error rate (WER) down to 0% for a clean spoken digit database, thus achieving state of the art performances.

5. Conclusion

We have reported an original writing of a delay dynamics through a signal processing approach, involving a convolution product description instead of the usual delay differential equation. The temporal impulse response attached to the linear Fourier filter involved in the delayed feedback oscillator loop, was revealed as a key physical ingredient for such a convolution product description. This representation was particularly useful to identify the space-time analogy involved in two currrently investigated research topic for delay dynamics: Chimera states, and delay-based Reservoir computing. Our analytical derivation has shown its relevance in the interpretation of these two recent successful achievements. Work is in progress to further develop this theory, in order to better understand chimera states, as well as to optimize the computational capabilities of delay-based RC.

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