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External Input-Facilitated Onset of Chaos in Recurrent Neural Networks

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Abstract– We investigate the transition from a fixed point state to chaos in a recurrent neural network. We focus on the intensity of external input to the network and show that the transition occurs when the external input increases. On the other hand, the standard deviation of the external input seems to have little impact on the transition.

1. Introduction

Neural networks with random connections have been receiving increasing attention. In the framework known as "reservoir computing" (Jaeger, 2001; Maass et al., 2002), such networks are found computationally powerful due to their nonlinear nature, where the dynamics is chaotic. However, the dynamics should not deviate far from an ordered one, since neural networks are required to hold useful information temporarily. The transition between ordered and chaotic states in recurrent neural networks is therefore crucial for flexible real-time computation.

An interesting work by Rajan et al. (2010) showed that the ongoing chaos can be suppressed by periodic external input with large magnitude. Recently, Kadmon and Sompolinsky (2015) developed systematically the mean field theory for the transition between a fixed point and chaotic fluctuation in randomly connected networks which consist of multiple subnetworks. Their results revealed the critical role of the synaptic gain and the shape of input-output transfer function in the transition of the dynamics to chaos. For large external input, the transfer function they used also predicts a suppression of chaos due to saturation.

However, in physiological experiments (e.g. Stokes et al., 2013), there is often an increase in mean firing rates after the onset of a stimulus, accompanied by the change in neural variability (Churchland et al., 2010). This could be a sign of change in dynamics, induced by enhanced external input. It therefore remains as a question whether this type of transition is allowed in recurrent neural networks.

In this paper, we consider another often-used transfer function, and show that the increasing in external input could also facilitate the onset of chaos.

2. Network Model

We consider a local network, which consists of N_E excitatory and N_I inhibitory recurrently connected neurons. In addition, the network receives external inputs from another N_R excitatory neurons. Following the idea of balanced network (Vreeswijk and Sompolinsky, 1996), we assume that each neuron receives $C_E = 200$ excitatory and $C_I = 400$ inhibitory synapses on average, from local recurrent connections, and further $C_R = 200$ excitatory synapses from a remote region. Therefore, the number of synapses on each neuron is large, but sparse compared to the total number of neurons in the network. It is known that in a spiking model with such assumption, the excitatory and inhibitory inputs to individual neurons dynamically balance each other (Vreeswijk and Sompolinsky, 1996), rendering the spikes to be determined by stochastic fluctuations. Here we use a rate-based model, where the local dynamics of the i^{th} neuron is given by

$$\begin{cases} \dfrac{dx_i^k}{dt} = -\dfrac{x_i^k}{\tau_k} + \sum_{j=1}^{N_E} J_j^{kE} r_j^E - \sum_{j=1}^{N_I} J_j^{kI} r_j^I + \sum_{j=1}^{N_R} J_j^{kR} r_j^R, & (1) \\ r_i^k = f(x_i^k), k \in \{E, I\}, & (2) \end{cases}$$

where x_i^k is the activation of neuron and r_i^k is the corresponding firing rate. The synaptic strength J_j^{kE}, J_j^{kI}, and J_j^{kR} are either zero, which corresponds to non-connected neurons, or drawn independently from certain distributions, with mean and standard deviation (SD) μ^{kl} and $\sigma^{kl}, k \in \{E, I\}, l \in \{E, I, R\}$. We adopt the often-used transfer function $f(x) = \frac{1}{2}(1 + tanh(x/T))$, where T indicates the nonlinearity. We make further simplifications by assuming that $\tau_E = \tau_I = \tau = 10$, $\mu^{EE} = \mu^{IE} = \mu^{ER} = \mu^{IR} = J = 0.2, \mu^{EI} = \mu^{II} = gJ = 5 \cdot 0.2, \sigma^{kl} = \mu^{kl}$ (Amit and Brunel, 1997), and $T = 10$. The external inputs are assumed to be constant over time.

3. Methods and Results

Two different states are found in the dynamics (Figure 1). When the intensity of external input is weak, a stable non-zero fixed point exists in the firing rate space, where almost any initial distribution of firing rates will be attracted into this state. The configuration of firing rates in this state is determined by the specific configuration of synapses, and firing rates in the external input. Note that although this configuration is random due to the randomness in synapses and external input, its statistical properties are rather simple, and can be obtained self-consistently. When the intensity of the external input exceeds some critical value, however, the fixed point is destabilized and the network state exhibits deterministic chaos, a common scenario reported in balanced networks.

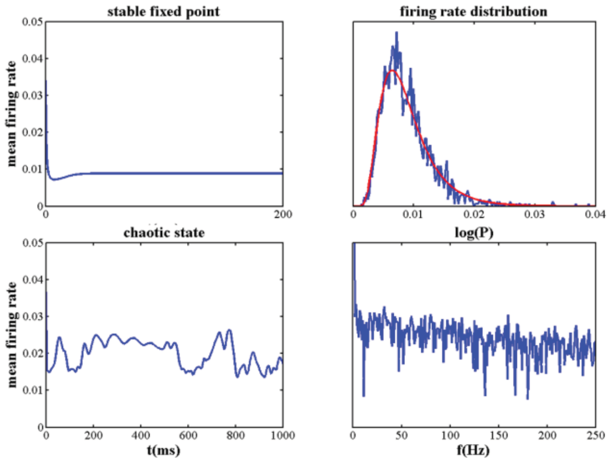

Figure 1. Two states in the dynamics. Top: the mean and distribution of firing rates in the fixed point state in a network with $N_E = 2000, N_I = 400, \mu_R = 0.02$, and $\sigma_R = 0.02$. The red curve indicates the distribution solved from the self-consistent equations. Bottom: the mean firing rate in a network with $\mu_R = 0.1, \sigma_R = 0.02$, and the logarithm of the power versus frequency for the chaotic state.

We first derive the self-consistent equations for the fixed point state by setting $\frac{dx_i^k}{dt} = 0$ for each neuron. Because the number of synapses on each neuron is large, the contribution of firing rates of individual neurons is small. Thus the state of each neuron is approximately independent with each other, and the summation terms on the r.h.s. of (1) can be taken as Gaussian noise. In the limit of large networks, the dynamics of each neuron is therefore given as follows:

$$\frac{dx}{dt} = -\frac{x}{\tau} + \eta, \tag{3}$$

where x, driven by a Gaussian term η, is also Gaussian. This observation allows us to describe the state simply with the mean μ_x and SD σ_x of x. The probability density of the firing rate is therefore

$$P(r) = \frac{1}{\sqrt{2\pi}\sigma_x} exp\left[-\frac{(f^{-1}(r) - \mu_x)^2}{2\sigma_x^2}\right] \cdot [f^{-1}(r)]'. \tag{4}$$

Under the assumption of the fixed point state, μ_x and σ_x can be obtained as

$$\begin{cases} \mu_x = \tau J[(C_E - gC_I)\mu_r + C_R\mu_R], & (5) \\ \sigma_x^2 = \tau^2 J^2 (C_E + g^2 C_I)(2\sigma_r^2 + \mu_r^2) & (6) \\ \qquad +\tau^2 J^2 C_R (2\sigma_R^2 + \mu_R^2), & \end{cases}$$

where μ_r and σ_r are the mean and SD of the firing rate, and the correlation between synaptic connection J_j^k and firing rate r_j^k vanishes again due to the large number of synapses.

Substituting (5), (6) into (4) and integrating for the first two moments of the distribution of firing rates provides following self-consistent equations:

$$\begin{cases} \mu_r = \int_0^1 rP(r, \mu_r, \sigma_r)\, dr, & (7) \\ \mu_r^2 + \sigma_r^2 = \int_0^1 r^2\, P(r, \mu_r, \sigma_r) dr. & (8) \end{cases}$$

In the fixed point state, the mean and SD of the firing rate distribution can be solved from these equations by numerical methods.

However, the chaotic state is much more complicated than what can be described by a low-dimensional system. We therefore wish to understand this transition to chaos from below, by continuously changing a bifurcation parameter in the fixed point state. To this end, we solve the self-consistent equations (7) and (8) at certain parameter values, and then use a continuation method to investigate the changes in solutions w.r.t. the intensity of the external input. The result is shown in Figure 2. Solutions for either the mean or SD of the firing rate distribution exist for finite μ_R, indicating a critical value at which the transition occurs.

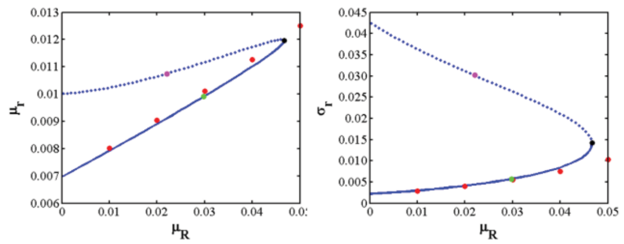

Figure 2. Dependency of mean and SD of the firing rate on mean external input. The solid line and dashed line indicate stable and unstable fixed points, respectively. The red points are simulation results in a network with $N_E = 2000, N_I = 400$, and $\sigma_R = 0$. The green, black and magenta points denote the solutions used to calculate the spectrum in Figure 3. Above the critical value, no fixed point exists in the dynamics. The outlier red point at $\mu_R = 0.05$ is a result due to the finite network used in the simulation.

The stability of fixed point state is determined by the Jacobian matrix of (1):

$$\mathfrak{J} = \left(J_{ij} f'(x_j) - \frac{1}{\tau}\delta_{ij}\right)$$

$$= J \cdot \begin{pmatrix} f'(x_1) & & \\ & \cdots & \\ & & f'(x_{N_E+N_I}) \end{pmatrix} - \frac{1}{\tau} I_{(N_E+N_I)\times(N_E+N_I)}, \tag{9}$$

where the first N_E columns in J correspond to synapses from excitatory neurons, the remaining N_I columns correspond to synapses from inhibitory neurons, and $\{x_i\}$

arranged accordingly. The fixed point is stable when all eigenvalues of this matrix have negative real parts. Note that J is a random matrix whose non-zero entries are drawn from two different distributions. The spectrum of such matrices have been studied in Rajan et al. (2006). Here we simply use numerical results to show the changes in it. In order to do this, we first solve μ_x and σ_x for given parameter values, and then sample a sufficiently large matrix according to these statistics and calculate the eigenvalues. The results are shown in Figure 3. Due to the correlation in the entries, the spectrum appears as an ellipse. As the solution of the fixed point moves along the blue curve in Figure 2, both the SD of the firing rate and the radius of the spectrum of the Jacobian matrix increase gradually. The bifurcation emerges when one of the eigenvalues crosses the imaginary axis. After that, the starting fluctuations further add to the SD of the firing rate, resulting in avalanche which forces the system quickly into chaotic dynamics.

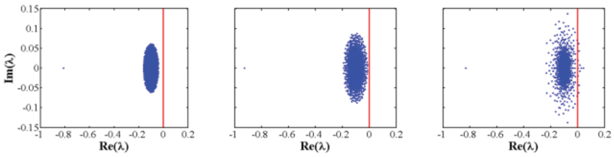

Figure 3. Changes in the spectrum of the Jacobian matrix. From left to right, calculated at the green ($\mu_R = 0.0298, \mu_r = 0.0099, \sigma_r = 0.0057$), the black ($\mu_R = 0.0467, \mu_r = 0.0120, \sigma_r = 0.0142$) and the magenta ($\mu_R = 0.0220, \mu_r = 0.0108, \sigma_r = 0.0302$) points in Figure 2, with $N_E = 10000$, and $N_I = 2000$.

Finally, we show that the SD of the external input has relatively small impact on this transition. We generate a network randomly with $N_E = 2000$, and $N_I = 400$, and then run the simulation in this network with external inputs of different means and SDs. The result is shown in Figure 4. Each square denotes the resulting variance in the state, with corresponding combination of μ_R and σ_R and random initial conditions. Therefore the zero variance indicates the fixed point state, and the transition can be found only for large μ_R.

4. Discussion

In this paper, we studied the transition from the fixed point state to chaos in a recurrent neural network, which is facilitated by an increasing external input. In the fixed point state, we found that the state can be characterized by two statistics that can be solved self-consistently. We showed how the transition occurs numerically, by continuously changing the bifurcation parameter. We also found that the SD of the external input seems not to largely influence the transition.

These results have some interesting implications. First of all, since the transition is independent of the SD of the external input, the degree of freedom in the configuration of the external input could be exploited for information coding. For example, an enhanced external input at the onset of stimulus sets chaos onset, encouraging the trajectory in network dynamics to explore misaligned dimensions, which has been shown crucial in many cognitive tasks (e.g. Raposo et al., 2014; Kaufman et al., 2014). Meanwhile, the information about the initial state is preserved through the specific configuration in the external input, thus the trajectory could either recover the initial state, or converge to some state corresponding to integrated information after the task. On the other hand, as indicated by the outlier red points in Figure 2, the transition to chaos is postponed in a finite network. This is due to the random fluctuation in the network structure, and is sometimes referred to as the quenched noise. This randomness implies that each realistic neural network may have a different coding scheme.

To realize certain cognitive functions, cortical networks are required quite often to dynamically switch between different types of dynamics: a noise-resistant one for information holding and a perturbation-sensitive one for information manipulation. The transition we studied here might therefore provide some insight on how this could be achieved in a recurrent neural network.

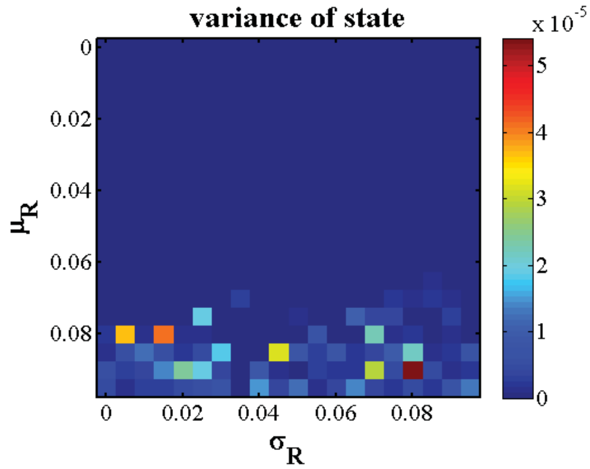

Figure 4. The SD of the external input has minor effect on the transition. The variance of the resulting state is shown by color.

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