2016 International Symposium on Nonlinear Theory and Its Applications,
NOLTA2016, Yugawara, Japan, November 27th-30th, 2016

Shishi-odoshi and large deviations

Syuji Miyazaki† and Kenta Yamada‡

†Graduate School of Informatics, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan
‡Kyoto-City Horikawa High School, Shibou-Horikawa-Machi 622-2, Nakagyo-ku, Kyoto 604-8254, Japan
†‡ELCAS Kyoto University, http://elcas.kyoto-u.ac.jp/
Email: syuji@acs.i.kyoto-u.ac.jp

Abstract—*Shishi-odoshi* is a traditional device found in Japanese gardens; composed of a bamboo tube that when filled with water revolves to empty and makes a clanking sound It consists of a water-filled bamboo tube which clacks against a stone when emptied, and the clack scars beasts and birds from gardens. For a fluctuating flow rate, intervals between the clacks distribute. The flow rate per unit time and the distribution function of the clack interval can be respectively identified as a velocity of a random walker and a first passage time distribution. The rate function of the flow rate per unit time is derived not according to its definition but by use of the distribution function of a first passage time. This idea is illustrated by coin-tossing large-deviation statistics.

1. Introduction

Shishi-odoshi is a water-filled hydraulic bamboo clapper against a stone when emptied. It is a simple device to drive away birds and animals, which makes a sound by water falling down. Its examples are illustrated in [1]. In this paper, we assume that the flow rate or the amount of water pouring into *Shishi-odoshi* per unit time fluctuates. And we discuss a relationship of the distribution of intervals between the clacks to large deviations of the flow rate per unit time. Although our basic idea was first described in [2] in the context of fluctuations of the flow rate, it can be generalized in a sense that the rate function of a random or chaotic variable can be derived from its *first-passage-time* problem.

2. Formalism

In this section, the common formalism of the previous [3] and the present studies is described.

Let V be a volume of *Shishi-odoshi*'s water container. The time-dependent flow rate per unit time is denoted as $f(t)$. At $t = t_0$ we start to pour water to the *Shishi-odoshi*, and it is filled at $t = t_0 + n$. In this case, the relation $\int_{t_0}^{t_0+n} f(t)\mathrm{d}x = V$ is satisfied, in which n is an interval between the clacks of *Shishi-odoshi*. In the following, an ideal *Shishi-odoshi* is considered, which discharges instantaneously a total amount of water at full level. One may regard f, V and n respectively as a velocity of a random walker starting from the origin, a distant goal and a first passage time to the goal is reached. Thus, measuring the intervals between the clacks of *Shishi-odoshi*, we can construct a distribution of the first passage time.

The local average z of the flow rate per unit time is given by

$$z = \frac{\int_{t_0}^{t_0+n} f(t)\mathrm{d}x}{n} = \frac{V}{n}.$$

The first passage times n distribute. So do the local averages z due to the above relation. The distribution of z depending on n is denoted as $P(n, z)$, from which we can obtain large deviation statistics of the flow rate per unit time. If n is much larger than its average auto-correlation time of $f(t)$, $P(n, z)$ is scaled as $P(n, z) = P(n, \overline{z}) \exp[-n\psi(z)]$, in which $P(n, \overline{z})$ is an algebraic factor depending on n and $\psi(z)$ is called rate function of the flow rate per unit time [4]. Let $\overline{z}$ be the long-time average as z. The rate function is concave up, which satisfies $\psi(z)|_{z=\overline{z}} = \left.\frac{d\psi(z)}{dz}\right|_{z=\overline{z}} = 0$. As a consequence of the central limit theorem, the rate function is quadratic around $z = \overline{z}$.

In our novel viewpoint inspired by the *shishi-odoshi*, we observe not directly the local average z or its instantaneous value of the flow rate per unit time but the first passage time n corresponding to the interval between the clacks in the case of *shishi-odoshi*. The distribution $P(n, z)$ of z can be regarded as a distribution $Q(V, n)$ of n via the relation $z = V/n$.

The transformation of variable from z to $V = nz$ satisfies the conservation of probability $P(n, z)dz = Q(V, n)dV$, so that we have

$$P(n, z) = Q(V, n)\frac{dV}{dz} = nQ(V, n),$$

$$P(n, \overline{z}) = \overline{n}Q(V, \overline{n}),$$

and the rate function $\psi(z)$ can be indirectly estimated as

$$-\frac{1}{n}\log\frac{nQ(V, n)}{\overline{n}Q(V, \overline{n})}$$

plotted against $z = V/n$, where $\overline{n} = V/\overline{z}$ is the long time average of the first passage time.

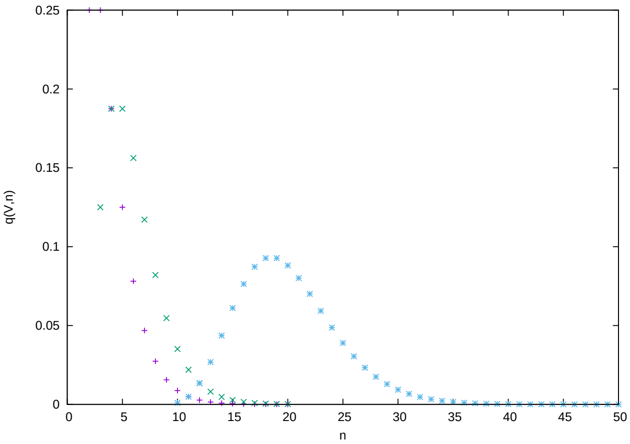

Figure 1: The first passage time distributions $q(V, n)$ plotted against n for $V = 2$ (+), 3 (×) and 10 (∗)

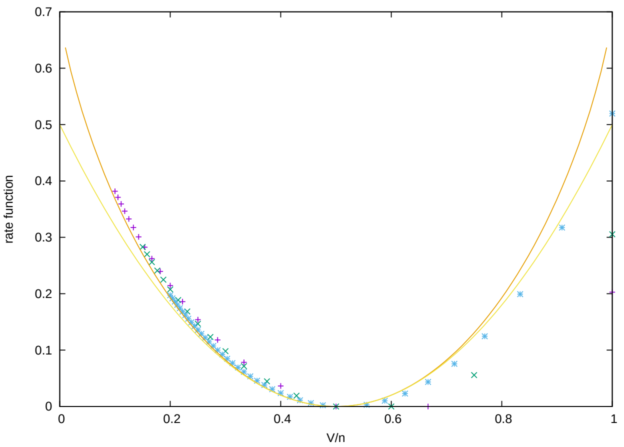

Figure 2: The approximate rate functions $-\frac{1}{n}\log\frac{nq(V,n)}{2Vq(V,2V)}$ plotted against V/n for $V = 2$ (+), 3 (×) and 10 (∗). The exact rate function (upper line) and the parabola coming from the central limiting theorem (lower line) are also drawn.

3. Discussion based on a concrete example

A concrete example is described in the following. An event that a waterdrop falls or does not fall is assumed to occur at regular unit intervals with equal probability, say, according to a fair coin tossing. In this case, f is a binary variable 0 or 1 depending on a integer-valued time step. Note that the water dropping interval in the real dripping faucet is strongly related to each volume of successive waterdrops which may be called the flow rate in this case [5]. The probability $r(n, z)$ that the head appears nz times in n time steps, equivalently the probability that a waterdrop falls nz times in n time steps, yielding the flow rate per unit time $z = \frac{nz}{n}$, is given by $r(n, z) = \frac{{}_nC_{nz}}{2^n} = \frac{n!}{(nz)!(n-nz)!2^n}$, which can be expressed by V instead of z as $r(n, V/n) = \frac{n!}{(V)!(n-V)!2^n}$. The probability $p(n, V/n)$ that the water container with volume V is filled exactly at time step n is given by $p(n, \frac{V}{n}) = \frac{1}{2}r(n-1, \frac{V-1}{n-1}) = \frac{(n-1)!}{(V-1)!(n-V)!2^n} = q(V, n)$. In Fig. 1, n-dependences of this probability are plotted for $V = 2$, 3 and 10. Note that both $P(n, z)$ and $Q(V, n)$ in the preceding section are probability densities and that both $r(n, z)$ and $q(V, n)$ in this section are not probability densities but probabilities.

Taking a large-container limit $V \to \infty$, we apply Stirling's formula $\log N! \sim N \log N - N$ to the factorials, so that we have the rate function $-\frac{1}{n}\log p(n, \frac{V}{n}) = \psi(z) = z\log z + (1-z)\log(1-z) + \log 2$ with $z = \frac{V}{n}$. At the long-time average $z = \bar{z} = 1/2$, the relations $\psi(z)|_{z=\bar{z}} = \left.\frac{d\psi(z)}{dz}\right|_{z=\bar{z}} = 0$ are satisfied. In the neighborhood of $z = \bar{z}$, $\psi(z)$ is approximated by the parabola $\psi(z) = 2\left(z - \frac{1}{2}\right)^2$, which implies the central limiting theorem.

The rate function of the flow rate per unit time can be estimated as

$$-\frac{1}{n}\log\frac{nq(V,n)}{2Vq(V,2V)}$$

plotted against $z = V/n$, where $\bar{n} = V/\bar{z} = 2V$ is the long time average of the first passage time, which is shown in Fig. 2 for small-container cases $V = 2$ (+), 3 (×) and 10 (∗) in comparison with the large-container limit (upper line) and the parabola indicating the central limit theorem (lower line). Although the latter holds only around the long time average, it is also drawn outward from the range in application of the central limiting theorem for eye guidance. In spite of small-container cases, a relatively good agreement is observed with the large-container limit. A systematic discrepancy is assumed to come from the fact that the first passage time distribution in a small-container case is asymmetric around the maximum, which eventually becomes symmetic in a large-container limit, as shown in Fig. 1.

4. Concluding remarks

It is difficult to observe fluctuating flow rates in a realistic system, as we mentioned in our preceding study [3].

Our indirect derivation of the rate function from the distribution of the first passage time without observing the instantaneous value $f(t)$ and its local average z can be applied to any stationary fluctuation of $f(t)$, although we confined ourselves to large deviations of the flow rate per unit time inspired by *shishi-odoshi*. One may regard the relation that the sum of a random variable V is equal to the local average z multiplied by the time span n for coarse-graining as the relation that the distance V is equal to the local average of a random velocity z multiplied by the first passage time n.

The concrete model described in the preceding section can also be regarded as a one-directional random walk, where the random walker either stops or jumps in a positive direction. A straightforward application to various mathematical or numerical models and experiments describing deterministic chaotic diffusion, where the large deviation statistics are approximately obtained from the first-passage-time distributions.

Acknowledgements

We thank Hiroki Hata for illuminating discussions. Financial support from Kyoto University ELCAS program for high school students is deeply appreciated.

References

[1] `https://en.wikipedia.org/wiki/Shishi-odoshi`

[2] Yamada, K. and Miyazaki, S. (2016) *Shishi-odoshi* and large deviations of flow rates per unit time, ELCAS Journal (Kyoto University) Vol. 1, No. 1, pp. 73–75.

[3] Yamada, K. and Miyazaki, S. (2016), *Large deviation statistics derived from first passage times — inspired by shishi-odoshi* —, FORMA Vol. 31, No. 1, pp.11-13.

[4] H. Mori and Y. Kuramoto, *Dissipative Structures and Chaos*, Springer (1998); its original Japanese text *San'itsukozo to kaosu*, Iwanami-Shoten (2000) is also available.

[5] Robert Shaw, *The Dripping Faucet as a Model Chaotic System*, Aerial Press, (1984).