2016 International Symposium on Nonlinear Theory and Its Applications,
NOLTA2016, Yugawara, Japan, November 27th-30th, 2016

Analysis of a 3-dimensional piecewise-constant chaos generator without constraint

Shuntaro Hamatani† and Tadashi Tsubone‡

†Department of Electrical, Electronics and Information Engineering, Nagaoka University of Technology,
1603-1 Kamitomioka, Nagaoka, 940-2137, Japan
Email: s143171@stn.nagaokaut.ac.jp, tsubone@vos.nagaokaut.ac.jp

Abstract—In this study, we propose an novel autonomous 3-dimensional piecewise-constant dynamical system without constraint, and analyze the nonliner phenomena by using two-dimensional(2-D) return map. The return map is derived rigorously and is represented by explicit expressions. In addition, some experimental results are obtained in the extremely simple circuit.

1. Introduction

The piecewise-constant systems[1] exhibit many nonliner phenomena, chaos, bifurcation and so on, in spite of the simple dynamics. The systems can be analysed with comparatice ease, because the piecewise-solutions are linear and the connections of the solutions are described explitly on the boundary[2]. Therefore, the piecewise-constant systems are useful for demonstrating nonlinear phenomena. For example, regorous analysis of quasiperiodic bifurcation in piecewise-constant systems with external forces have been reported[3][4], and analysis of synchronization in a coupled piecewice-constant system have been discussed [4]. In addition, methods to derive rigorous solutions have been developed for bath autonomous and non-autonomous piecewice-constant systems[4][5]. However, previous piecewise-constant circuits are discribed by constrained equations, that is, state variables are constrained in partial hyperplanes of phase space depended on some conditions. Therefore, some considerations are insufficient about more natural systems without constraint. In this study, we present a novel three-dimensional(3-D) autonomous piecewise-constant chaos generator, and analysis the chaotic behavior by using 2-D return map. The results implies chaos generation. Some theoretical results are confirmed in laboratory.

2. 3-D chaos-generationg piecewise-constant oscilator

Figure 1 shows the circuit diagram of the piecewise-constant circuit. This circuit consists of six voltage-controlled current sources(VCCSs) that has signum characteristic and three capacitors. These VCCSs are realized by operational transconductance amplifiers. In order to describe the dynamics, we define two functions as follows:

$$\mathrm{Sgn}(x) = \begin{cases} 1 \text{ for } x \geq 0 \\ -1 \text{ for } x < 0 \end{cases}, U(x) = \begin{cases} 1 \text{ for } x \geq 0 \\ 0 \text{ for } x < 0. \end{cases} \quad (1)$$

Then, the circuit dynamics is described as follows:

$$\begin{cases} \dot{x} = \mathrm{Sgn}(y-1) \\ \dot{y} = -\mathrm{Sgn}(x) - b \cdot \mathrm{Sgn}(y) + a \cdot U(z) \cdot \mathrm{Sgn}(y) \\ \dot{z} = -\mathrm{Sgn}(x). \end{cases} \quad (2)$$

where " · "represents the derivative of τ and the following normalized variables and parameters are used :

$$\tau = \frac{I_s}{C_3 E}t,\ x = \frac{C_1}{C_3 E}v_1,\quad y = \frac{C_1}{C_2 E}v_2,\quad z = \frac{1}{E}v_3,$$
$$a = \frac{I_a}{I_s},\quad b = \frac{I_b}{I_s}. \quad (3)$$

I_s, I_a,and I_b are bias currents of operational transconductance amplifiers that are represented by trapezoid symbols in Fig.1.

The dynamics are represented by twelve local vector fields and the conditions. These are shown in Table 1.

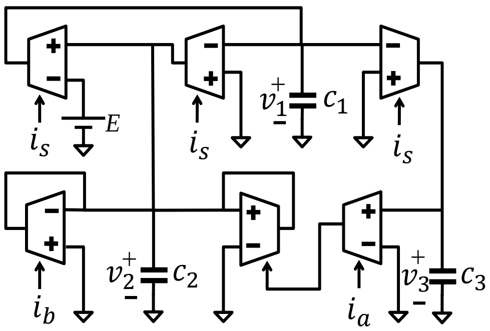
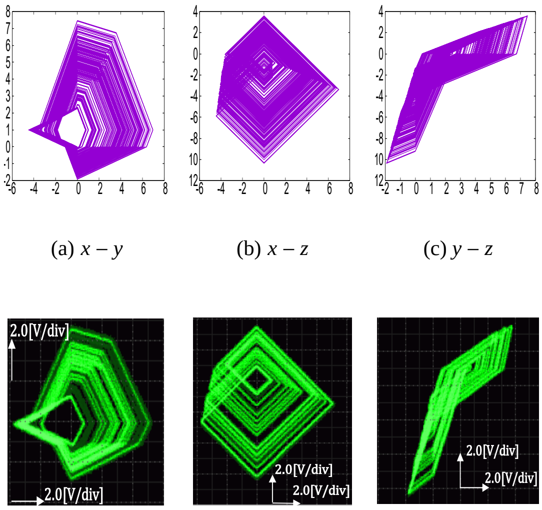


Fig 1: Implemented circuit ($x \propto v_1$, $y \propto v_2$, $z \propto v_3$)



(a) $x - y$ (b) $x - z$ (c) $y - z$

(d) $v_1 - v_2$ (e) $v_1 - v_3$ (f) $v_2 - v_3$

Fig 2: Typical chaotic attractor (a=1.5, b=0.7) (a)(b)(c) Regorous solutions , (d)(e)(f) Laboratory measurements 2.0 [V/div.]

Figure 2 shows a typical chaotic attractor with $a = 1.5, b = 0.7$. We measured same attactor in laboratory.

3. 2-D return map

In order to analyze the chaotic behavior, we focus on parameter $a = 1.5, b = 0.4$ for simplicity and derive 2-D return map. First, we define the domain S:

$$S = \{\boldsymbol{x} = (x, y, z)\} \quad | \quad y = 1\}. \tag{4}$$

The trajectory starting from $\boldsymbol{x_0}$ on S must return to $\boldsymbol{x_1}$ on S. Then, we can define a 2-D return map $\boldsymbol{F}$ from S to itself.

$$\boldsymbol{F} : S \rightarrow S, (x_1, z_1) = \boldsymbol{F}_i(x_0, z_0)$$
$$= (f_i(x_0, z_0), g_i(x_0, z_0)) \quad (i = 0, 1, 2, ..., 7). \tag{5}$$

There are eight kind of trajectories that from S to itself. The trajectories are derived by the thresholds that are given as follows:

$$Th_0 = \frac{b^2 - 1}{2b \cdot (b + 1)} \cdot z_0, Th_1 = \frac{1}{1 + b},$$
$$Th_2 = \frac{\left(b^3 + b^2 - b - 1\right) \cdot z_0 - 4 \cdot b}{2 \cdot b^3 - 2 \cdot b}. \tag{6}$$

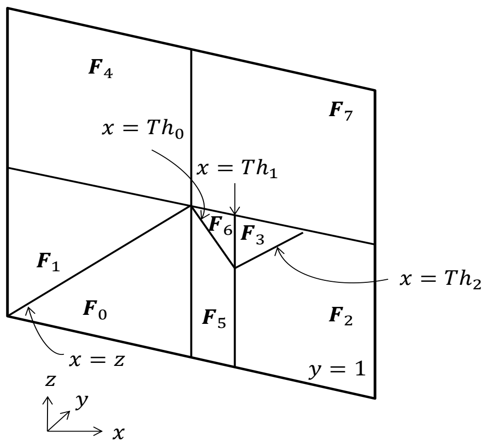


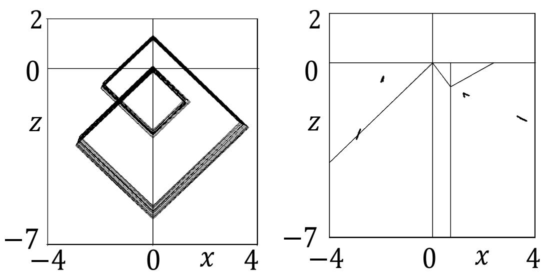
Fig 3: Domain S

Figure 3 shows the local regions of maps $\boldsymbol{F}_i$. By using solution of (2), 2-D return map is derived rigorously and is represented by explicit expressions.

$$(x_1, z_1) = \boldsymbol{F}_i(x_0, z_0)$$

$$
= \begin{cases}
\boldsymbol{F}_0(x_0, z_0) & \text{for} \quad x_0 \geq z_0, x_0 < 0, z_0 < 0, \\
\boldsymbol{F}_1(x_0, z_0) & \text{for} \quad x_0 < z_0, x_0 < 0, z_0 < 0, \\
\boldsymbol{F}_2(x_0, z_0) & \text{for} \quad x_0 \geq Th_2, x_0 \geq Th_1, z_0 < 0, \\
\boldsymbol{F}_3(x_0, z_0) & \text{for} \quad x_0 < Th_2, x_0 \geq Th_1, z_0 < 0, \\
\boldsymbol{F}_4(x_0, z_0) & \text{for} \quad x_0 < 0, z_0 \geq 0, \\
\boldsymbol{F}_5(x_0, z_0) & \text{for} \quad x_0 \geq Th_0, 0 \leq x_0 < Th_1, \\
\boldsymbol{F}_6(x_0, z_0) & \text{for} \quad x_0 < Th_0, 0 \leq x_0 < Th_1, \\
\boldsymbol{F}_7(x_0, z_0) & \text{for} \quad x_0 \geq 0, z_0 \geq 0,
\end{cases} \tag{7}
$$

where (f_i, g_i) are shown in Table 2.



(a) $x - z$ (b) $x - z$

Fig 4: Chaotic attractor (a), 2-D Return map (b), $a = 1.5, b = 0.4$.

Figure 4 is chaotic attractor and corresponding 2-D return map. The behavior of system without transients is governed by only $\boldsymbol{F}_0, \boldsymbol{F}_1$, and $\boldsymbol{F}_2$. Therefore, discussion about stability of dynamics can be considered by $\boldsymbol{F}_0, \boldsymbol{F}_1$, and $\boldsymbol{F}_2$. The stability can be analyzed by using eigenvalues of Jacobian matrix of 2-D return map. Each Jacobian matrices are explicity given as follows,

$$
\boldsymbol{DF}_0(x, z) = \begin{bmatrix} -\frac{b-1}{-b-1} & 0 \\ \frac{b-1}{-b-1} - 1 & 1 \end{bmatrix},
$$

for $\quad x_0 \geq z_0, x_0 < 0, z_0 < 0,$

$$
\boldsymbol{DF}_1(x, y) = \begin{bmatrix} -\frac{2b-2a}{-b-1} - 1 & 1 - \frac{-b+2a-1}{-b-1} \\ \frac{2b-2a}{-b-1} & \frac{-b+2a-1}{-b-1} \end{bmatrix},
$$

for $\quad x_0 < z_0, x_0 < 0, z_0 < 0,$

$$
\boldsymbol{DF}_2(x, y) = \begin{bmatrix} \frac{b-1}{b+1} & 0 \\ -\frac{b-1}{b+1} - 1 & 1 \end{bmatrix},
$$

for $\quad x_0 \geq Th_2, x_0 \geq Th_1, z_0 < 0.$ (8)

By using these Jacobian matrices, we obtained the eigenvalues of Jacobian matrix of the $n-$ times composition map for large n theoretically. Since any one of the absolute values of eigenvalues is $|\lambda| \geq 1$, we can say that the attractor shown in Fig. 4 is unstable.

4. Conclusion

We realized 3-D autonomous piecewise-constant chaos generator without constraint. Using 2-D return map, we confirmed the unstability of the chaotic behavior. In the future, we will clarify the existence region of chaotic attractor.

References

[1] T. Tsubone and T. Saito, "Manifold Piecewice Constant Systems and Chaos," IEICE Trans. Fundamentals, vol. E82-A, no. 8, pp. 1619-1626, (1999).

[2] T. Tsubone, T. Saito and N. Inaba, "Design of an analog chaos-generating circuit using piecewise-constant dynamics," Prog. Theor. Exp. Phys. 2016, 053A01.

[3] T. Tsubone, N. Inaba, T. Tsubouchi, T.Yoshinaga, "Synchronization Phenomena from an Extremely Simplified Piecewise-Constant Driven Oscillator," IEICE Trans. A (Japanese Edition), vol.J93-A, No,5, pp.375-383, 2010.

[4] H. Kuriyama, T. Tsubone, "Arnold tongues of 2-D Piecewise Constant Driven Oscillator," Proc. NOLTA, B2L-D5-6237, pp. 479-482, 2015.

[5] K. Suzuki, T. Tsubone, "In-Phase and Anti-Phase Syncronization Phenomena in Coupled Systems of Piecewice Constant Oscilators," IEICE, Trans. Fundamentals, Vol. E98-A, No.1, pp340-353, 2015.

Table 1: Local vector fields and regions for k

k	$\alpha(k)$	D_k
0	${}^t\left(-1, \quad 1+b, \quad 1\right)$	$\{\boldsymbol{x} \mid x < 0, y < 0, y-1 < 0, z < 0\}$
1	${}^t\left(-1, \quad -1+b, \quad -1\right)$	$\{\boldsymbol{x} \mid x \geq 0, y < 0, y-1 < 0, z < 0\}$
2	${}^t\left(-1, \quad 1+b, \quad 1\right)$	$\{\boldsymbol{x} \mid x < 0, y \geq 0, y-1 < 0, z < 0\}$
3	${}^t\left(-1, \quad -1-b, \quad -1\right)$	$\{\boldsymbol{x} \mid x \geq 0, y \geq 0, y-1 < 0, z < 0\}$
4	${}^t\left(1, \quad 1-b, \quad 1\right)$	$\{\boldsymbol{x} \mid x < 0, y \geq 0, y-1 \geq 0, z < 0\}$
5	${}^t\left(1, \quad -1-b, \quad -1\right)$	$\{\boldsymbol{x} \mid x \geq 0, y \geq 0, y-1 \geq 0, z < 0\}$
6	${}^t\left(-1, \quad 1+b-a, \quad 1\right)$	$\{\boldsymbol{x} \mid x < 0, y < 0, y-1 < 0, z \geq 0\}$
7	${}^t\left(-1, \quad -1+b-a, \quad -1\right)$	$\{\boldsymbol{x} \mid x \geq 0, y < 0, y-1 < 0, z \geq 0\}$
8	${}^t\left(-1, \quad 1-b+a, \quad 1\right)$	$\{\boldsymbol{x} \mid x < 0, y \geq 0, y-1 < 0, z \geq 0\}$
9	${}^t\left(-1, \quad -1-b+a, \quad -1\right)$	$\{\boldsymbol{x} \mid x \geq 0, y \geq 0, y-1 < 0, z \geq 0\}$
10	${}^t\left(-1, \quad 1-b+a, \quad 1\right)$	$\{\boldsymbol{x} \mid x < 0, y \geq 0, y-1 \geq 0, z \geq 0\}$
11	${}^t\left(-1, \quad -1-b+a, \quad -1\right)$	$\{\boldsymbol{x} \mid x \geq 0, y \geq 0, y-1 \geq 0, z \geq 0\}$

Table 2: piecewise-linear 2-D maps for i

i	f_i	g_i
0	$\frac{1}{-b-1} - \frac{(b-1)\cdot x_0+1}{-b-1}$	$z_0 + \frac{(b-1)\cdot x_0+1}{-b-1} - x_0 - \frac{1}{-b-1}$
1	$\frac{2\cdot a\cdot z_0+b\cdot x_0-2\cdot a\cdot x_0-x_0}{b+1}$	$\frac{b\cdot z_0-2\cdot a\cdot z_0+z_0-2\cdot b\cdot x_0+2\cdot a\cdot x_0}{b+1}$
2	$\frac{(b-1)\cdot\left(x_0+\frac{1}{-b-1}\right)}{b+1} - \frac{1}{1-b}$	$z_0 - \frac{(b-1)\cdot\left(x_0+\frac{1}{-b-1}\right)}{b+1} - x_0 + \frac{1}{1-b}$
3	$\frac{(b-1)\cdot\left(z_0-\frac{(b-1)\cdot\left(x_0+\frac{1}{-b-1}\right)}{b+1}-x_0\right)}{b-a+1} + z_0 - x_0 - \frac{1}{b-a+1}$	$\frac{1}{b-a+1} - \frac{(b-1)\cdot\left(z_0-\frac{(b-1)\cdot\left(x_0+\frac{1}{-b-1}\right)}{b+1}-x_0\right)}{b-a+1}$
4	$-\frac{(-b+a-1)\cdot(z_0-x_0)+(b-a-1)\cdot x_0+1}{-b-1} + z_0 - x_0 + \frac{1}{-b-1}$	$\frac{(-b+a-1)\cdot(z_0-x_0)+(b-a-1)\cdot x_0+1}{-b-1} - \frac{1}{-b-1}$
5	$\frac{(-b-1)\cdot x_0+1}{1-b} - \frac{1}{1-b}$	$z_0 - \frac{(-b-1)\cdot x_0+1}{1-b} - x_0 + \frac{1}{1-b}$
6	$\frac{(b-1)\cdot(z_0-x_0)+(-b-1)\cdot x_0+1}{-b+a+1} + z_0 - x_0 - \frac{1}{-b+a+1}$	$\frac{1}{-b+a+1} - \frac{(b-1)\cdot(z_0-x_0)+(-b-1)\cdot x_0+1}{-b+a+1}$
7	$-\frac{(-b+a-1)\cdot z_0+1}{-b-1} + z_0 + x_0 + \frac{1}{-b-1}$	$\frac{(-b+a-1)\cdot z_0+1}{-b-1} - \frac{1}{-b-1}$