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Stability analysis of amplitude death in Cartesian product networks of delay-coupled Landau-Stuart oscillators

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Abstract—This report investigates amplitude death in Cartesian product networks of delay-coupled oscillators. The Cartesian product networks consist of two sub-networks which have different connection delays with each other; that is, the connection delays are not identical in the whole networks. Although such networks are difficult to analyze, the feature of the Cartesian product networks allows us to analyze such networks easily. The analytical result is confirmed by numerical simulations.

1. Introduction

The various collective phenomena in coupled oscillators have been widely investigated in biological, physical, chemical, and social systems [1]. One of such phenomena is amplitude death where a homogeneous steady state in coupled oscillators is stabilized by diffusive connections; that is, the oscillations of all the oscillators are quenched. Although amplitude death never occurs in diffusively-coupled identical oscillators [2], connection delays can cause amplitude death even in coupled identical oscillators [3].

Amplitude death induced by the connection delays has been great interest in nonlinear science [4]. Various types of delay connections that cause amplitude death have been proposed such as the distributed-delay connection [5, 6], the multiple-delay connection [7], the time-varying delay connection [8], the integrated delay connection [9], the digital delay connection [10], the multicomponent delay connection [11], and the mixed time-delay connection [12]. All the previous studies assumes that all the connection delays between oscillators are identical in the whole network. In the real world, however, it is totally impractical that all the connection delays are identical. Generally, it is difficult to analyze the coupled oscillators with non-identical connection delays.

The Cartesian product is one of the basic operation on Graph theory [13]. By using Cartesian product, we can construct various complex networks from simpler sub-networks, for instance, regular grids are constructed from two path graphs. It is well-known that the eigenvalues of the Laplacian matrix of a Cartesian product network are calculated by the sum of the eigenvalues of its sub-networks. Based on this fact, some researchers have investigated partial and full synchronization in Cartesian product networks of coupled oscillators [14, 15, 16].

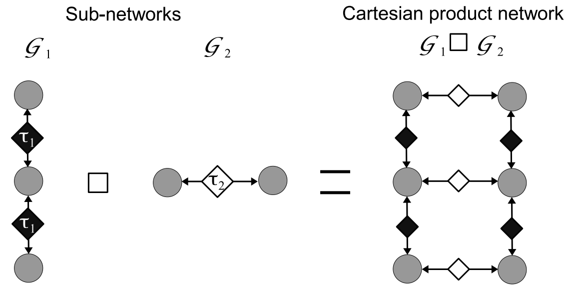


Figure 1: Illustration of Cartesian product network $\mathcal{G}_1 \square \mathcal{G}_2$ consisting of two sub-networks $\mathcal{G}_1$ and $\mathcal{G}_2$ which have different connection delays τ_1 and τ_2, respectively.

In this report, we investigate amplitude death in Cartesian product networks of delay-coupled oscillators. The Cartesian product networks consist of two sub-networks which have different connection delays with each other. Therefore, the connection delays are not identical in the whole networks. Even in such situation, the feature of Cartesian product allows us to easily analyze the stability of amplitude death. Furthermore, it is shown that the stability of amplitude death is heavily depends on the topology of the sub-networks.

The following notations are used throughout this report. $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is the graph consisting of set of nodes $\mathcal{V}$ and edges $\mathcal{E}$. Conversely, $\mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{G})$ represent the sets of nodes and edges of the graph $\mathcal{G}$, respectively. $\boldsymbol{A}_\mathcal{G}$ is the adjacency matrix of graph $\mathcal{G}$: if i-th and l-th nodes are connected by an edge, then $\{\boldsymbol{A}_\mathcal{G}\}_{il} = \{\boldsymbol{A}_\mathcal{G}\}_{li} = 1$; otherwise, $\{\boldsymbol{A}_\mathcal{G}\}_{il} = \{\boldsymbol{A}_\mathcal{G}\}_{il} = 0$. The matrix I_N denotes the $N \times N$ unit matrix. The imaginary unit is defined as $j := \sqrt{-1}$.

2. Cartesian Product network of delay coupled oscillators

This section briefly introduces the Cartesian product network consisting of two sub-networks. Then, we will explain the delayed coupled oscillators of the Cartesian product network where the two sub-networks have different connection delays.

2.1. Cartesian product network

The Cartesian product network consisting of two sub-networks $\mathcal{G}_1 = (\mathcal{V}(\mathcal{G}_1), \mathcal{E}(\mathcal{G}_1))$ and $\mathcal{G}_2 = (\mathcal{V}(\mathcal{G}_2), \mathcal{E}(\mathcal{G}_2))$ is denoted by $\mathcal{G}_1 \square \mathcal{G}_2$. The nodes set of $\mathcal{G}_1 \square \mathcal{G}_2$ is given by $V(\mathcal{G}_1 \square \mathcal{G}_2) = V(\mathcal{G}_1) \times V(\mathcal{G}_2)$. Edge $\left((v_1, v_2), (v_1', v_2')\right)$ is an edge in $\mathcal{E}(\mathcal{G}_1 \square \mathcal{G}_2)$ if $v_1 = v_1'$ and $(v_2, v_2') \in \mathcal{E}(\mathcal{G}_2)$ (or if $v_2 = v_2'$ and $(v_1, v_1') \in \mathcal{E}(\mathcal{G}_1)$). Figure 1 shows an example of the Cartesian product network. The adjacency matrix of the Cartesian product network $\mathcal{G}_1 \square \mathcal{G}_2$ is given by

$$\boldsymbol{A}_{\mathcal{G}_1 \square \mathcal{G}_2} = \boldsymbol{A}_{\mathcal{G}_1} \otimes \boldsymbol{I}_n + \boldsymbol{I}_m \otimes \boldsymbol{A}_{\mathcal{G}_2}, \tag{1}$$

where m and n are the number of nodes in sub-networks $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. The symbol $\otimes$ denotes the Kronecker product.

2.2. Delayed coupled oscillators

Let us consider the delayed coupled oscillators of the Cartesian product network $\mathcal{G}_1 \square \mathcal{G}_2$ illustrated in Fig. 1. The dynamics of the oscillators are given by,

$$\dot{Z}_i(t) = (a + j\omega - |Z_i(t)|^2)Z_i(t) + u_i^{(1)}(t) + u_i^{(2)}(t), \quad (i = 1, \ldots, mn), \tag{2}$$

where $Z_i(t) \in \mathbb{C}$ is the state variables of i-th oscillators. $a > 0$ and $\omega > 0$ respectively represent instability of the fixed point $Z_i^* = 0$ and the natural frequency of oscillators. Each oscillator receives the input signal $u_i^{(1)}(t)$ and $u_i^{(2)}(t)$ from sub-networks $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.

$$u_i^{(1)}(t) = k\left\{\frac{1}{d_i^{(1)}}\left(\sum_{l=1}^{mn} c_{i,l}^{(1)} Z_l(t-\tau_1)\right) - Z_i(t)\right\},$$
$$u_i^{(2)}(t) = k\left\{\frac{1}{d_i^{(2)}}\left(\sum_{l=1}^{mn} c_{i,l}^{(2)} Z_l(t-\tau_2)\right) - Z_i(t)\right\}, \tag{3}$$

where k is the coupling strength. τ_1 and τ_2 denote the connection delays in sub-networks $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively (see Fig. 1). Note that the connection delays would differ from sub-network to sub-network. $c_{i,l}^{(1)}$ and $c_{i,l}^{(2)}$ is (i, l) elements of adjacency matrix $\boldsymbol{A}_{\mathcal{G}_1} \otimes \boldsymbol{I}_n$ and $\boldsymbol{I}_m \otimes \boldsymbol{A}_{\mathcal{G}_2}$, respectively. $d_i^{(1),(2)}$ represent the degree of i-th oscillator in sub-networks $\mathcal{G}_1$ and $\mathcal{G}_2$. The coupled oscillators (2), (3) have the homogeneous steady state

$$[Z_1^*, \cdots, Z_{mn}^*]^T = [0, \cdots, 0]^T. \tag{4}$$

3. Linear stability analysis

Linearizing Eqs. (2) and (3) around steady state (4), we obtain

$$\dot{z}_i(t) = (a + j\omega - 2k)z_i(t) + \frac{k}{d_i^{(1)}} \sum_{l=1}^{mn} c_{i,l}^{(1)} z_l(t-\tau_1) + \frac{k}{d_i^{(2)}} \sum_{l=1}^{mn} c_{i,l}^{(2)} z_l(t-\tau_2), \tag{5}$$

where $z_i(t) := Z_i(t) - Z_i^*$ is the perturbation from steady state (4). Linear system (5) can be rewritten as

$$\dot{\boldsymbol{X}}(t) = (a + j\omega - 2k)\boldsymbol{X}(t) + k(\boldsymbol{E}_1 \otimes \boldsymbol{I}_n)\boldsymbol{X}(t-\tau_1) + k(\boldsymbol{I}_m \otimes \boldsymbol{E}_2)\boldsymbol{X}(t-\tau_2), \tag{6}$$

where $\boldsymbol{X}(t) := [z_1(t), \cdots, z_{mn}(t)]^T$. The matrices $\boldsymbol{E}_1 := \boldsymbol{D}_{\mathcal{G}_1}^{-1}\boldsymbol{A}_{\mathcal{G}_1}$ and $\boldsymbol{E}_2 := \boldsymbol{D}_{\mathcal{G}_2}^{-1}\boldsymbol{A}_{\mathcal{G}_2}$ denote the network topologies of sub-networks $\mathcal{G}_1$ and $\mathcal{G}_2$, where $\boldsymbol{D}_{\mathcal{G}_1} \in \mathbb{R}^{m\times m}$ ($\boldsymbol{D}_{\mathcal{G}_2} \in \mathbb{R}^{n\times n}$) is the diagonal matrix of oscillator's degree, i.e., its i-th diagonal element is the degree of i-th oscillator on sub-network $\mathcal{G}_1$ ($\mathcal{G}_2$).

The stability of linear system (6) is governed by the roots of the following characteristic equation.

$$G(s) := \det\left[(s - a - j\omega + 2k)\boldsymbol{I}_{nm} - k\left\{(\boldsymbol{E}_1 \otimes \boldsymbol{I}_n)e^{-s\tau_1} + (\boldsymbol{I}_m \otimes \boldsymbol{E}_2)e^{-s\tau_2}\right\}\right]. \tag{7}$$

It is known that the matrices $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ can be diagonalized as follows [17],

$$\boldsymbol{T}_1^{-1}\boldsymbol{E}_1\boldsymbol{T}_1 = \mathrm{diag}(\rho_1, \ldots, \rho_m),$$
$$\boldsymbol{T}_2^{-1}\boldsymbol{E}_2\boldsymbol{T}_2 = \mathrm{diag}(\sigma_1, \ldots, \sigma_n),$$

where $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are transformation matrices. $\rho_1, \ldots, \rho_m$ and $\sigma_1, \ldots, \sigma_n$ denote the eigenvalues of $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$, respectively. The matrices $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ in Eq. (7) can be simultaneously diagonalized by using the transformation matrix $(\boldsymbol{T}_1 \otimes \boldsymbol{T}_2)$ as follows:

$$\begin{aligned} G(s) &= \det\Big[(\boldsymbol{T}_1^{-1} \otimes \boldsymbol{T}_2^{-1}) \\ &\quad \{(s-a-j\omega+2k)\boldsymbol{I}_{nm} - k\{(\boldsymbol{E}_1 \otimes \boldsymbol{I}_n)e^{-s\tau_1} + (\boldsymbol{I}_m \otimes \boldsymbol{E}_2)e^{-s\tau_2}\}\} \\ &\quad (\boldsymbol{T}_1 \otimes \boldsymbol{T}_2)\Big] \\ &= \det\big[(s - a - j\omega + 2k)\boldsymbol{I}_{nm} - \\ &\quad k\{(\mathrm{diag}(\rho_1, \ldots, \rho_m) \otimes \boldsymbol{I}_n)\, e^{-s\tau_1} \\ &\quad + (\boldsymbol{I}_m \otimes \mathrm{diag}\,(\sigma_1, \ldots, \sigma_n))\, e^{-s\tau_2}\}\big]. \end{aligned}$$

This diagonalization allows us to separate the characteristic Eq. (7) into mn modes,

$$G(s) = \prod_{p=1}^{m}\left\{\prod_{q=1}^{n} g(s, \rho_p, \sigma_q)\right\}, \tag{8}$$

where,

$$g(s, \rho, \sigma) := s - a - j\omega + 2k - k(\rho e^{-s\tau_1} + \sigma e^{-s\tau_2}).$$

As a consequence, steady state (4) is stable if and only if all the mn modes of Eq. (8) is stable.

For checking the stability of Eq. (8), we will focus on roots of $g(s, \rho, \sigma) = 0$. The stability of Eq. (8) changes only when the roots crosses the imaginary axis. Substituting $s =$

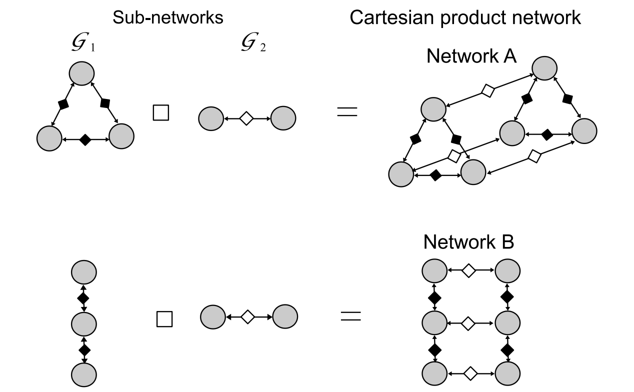


Figure 2: Two Cartesian product networks A and B

$j\lambda$ ($\lambda \in \mathbb{R}$) into $g(s,\rho,\sigma) = 0$ gives us the following two equations,

$$-a + 2k - k\rho\cos(\lambda\tau_1) - k\sigma\cos(\lambda\tau_2) = 0,$$
$$\lambda - \omega + k\rho\sin(\lambda\tau_1) + k\sigma\sin(\lambda\tau_2) = 0. \quad (9)$$

Solving Eq. (9) in terms of τ_1 and τ_2 yields the marginal stability curves on the connection parameter (τ_1, τ_2) space [7]. Moreover, in order to derive the stability region from the marginal stability curves, we have to check the direction of the roots crossing the imaginary axis. The direction can be checked by the real part of $\mathrm{d}s/\mathrm{d}\tau_2$ at $s = j\lambda$,

$$\mathrm{Re}\left[\frac{\mathrm{d}s}{\mathrm{d}\tau_2}\right]_{s=j\lambda} = \mathrm{Re}\left[-\frac{j\lambda k\sigma e^{-j\lambda\tau_2}}{1 + k\left(\rho\tau_1 e^{-j\lambda\tau_1} + \sigma\tau_2 e^{-j\lambda\tau_2}\right)}\right]. \quad (10)$$

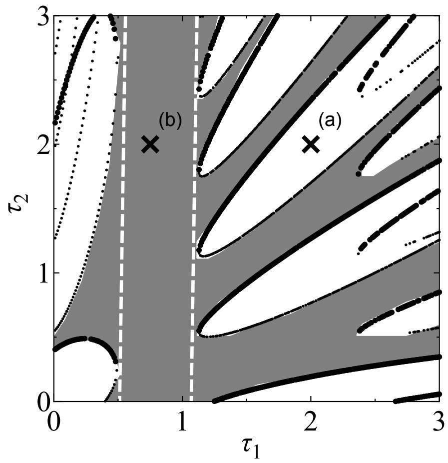
The positive (negative) sign of Eq. (10) denotes that the roots crossing the imaginary axis from left to right (right to left) as τ_2 increases.

4. Numerical examples

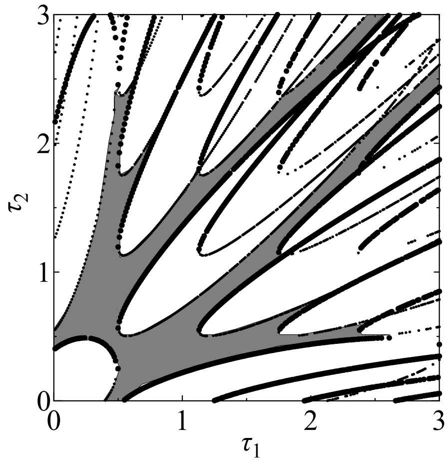
For numerical examples, we consider two Cartesian product networks A and B illustrated in Fig 2. Both of the networks have the same number of oscillators and same sub-network $\mathcal{G}_2$; that is, they have different sub-network $\mathcal{G}_1$. The eigenvalues of $\boldsymbol{E}_1$ for Network A are $\rho_{1,2} = -0.5$, $\rho_3 = 1$. Those for Network B are $\rho_1 = -1$, $\rho_2 = 0$, $\rho_3 = 1$. The eigenvalues of $\boldsymbol{E}_2$ are $\sigma_1 = -1$, $\sigma_2 = 1$. Throughout this report, the parameters of oscillators (2) and the coupling strength are fixed at

$$a = 0.50, \quad \omega = \pi, \quad k = 2.0. \quad (11)$$

Figure 3 shows the marginal stability curves and the stability region on the connection parameter (τ_1, τ_2) space. The thin (bold) curves denote that when a parameter set (τ_1, τ_2) crosses the curves with increasing τ_2, one root of $g(s,\rho,\sigma) = 0$ crosses the imaginary axis from left to right (right to left). The shaded area shows the stability region where all the roots of $G(s) = 0$ located on the left-half of the complex plane. In other words, the local stability of steady state (4) is guaranteed in this region.



(a) Network A



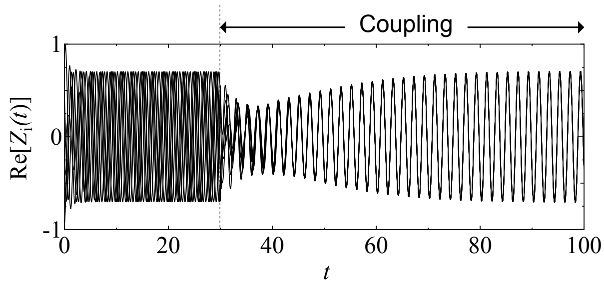
(b) Network B

Figure 3: The marginal stability curves (i.e., solid curves) and the stability region (i.e., shaded area) of Networks A and B illustrated in Fig. 2.

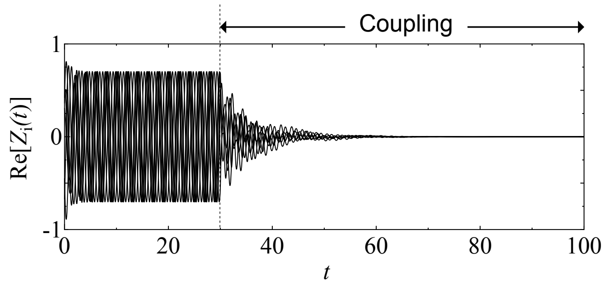
Comparing Fig. 3(a) with Fig. 3(b), the region for Network B is symmetry about the slanted line $\tau_1 = \tau_2$, while it is not symmetry for Network A. Moreover, the region for Network A has the range of τ_1, which is between two white dotted lines in Fig 3(a), such that we can use the long connection delay τ_2 of sub-network $\mathcal{G}_2$ to induce amplitude death [1]. It should be noted that long connection delays never induce amplitude death if the connection delays are identical (i.e., $\tau_1 = \tau_2$) in the whole networks [3].

Figure 4 shows the time-series data of the state variables $\mathrm{Re}[Z_i(t)]$ at points (a):$(\tau_1, \tau_2) = (2.0, 2.0)$ and (b):$(\tau_1, \tau_2) = (0.75, 2.0)$ in Fig. 3(a). At $t = 30$, all the oscillators are coupled. For point (a), the variables still oscillate after coupling. For point (b), they converge onto steady state (4).

[1] We can use even a diffusive connection (i.e., $\tau_2 = 0$).



(a) point (a): $(\tau_1, \tau_2) = (2.0, 2.0)$



(b) point (b): $(\tau_1, \tau_2) = (0.75, 2.0)$

Figure 4: Time-series data of $\mathrm{Re}[Z_i(t)]$ at point (a) and (b) in Fig. 3(a).

5. Conclusion

This report has investigated amplitude death in Cartesian product networks of delayed coupled oscillators, where two sub-networks of the Cartesian product networks have different connection delays with each other. By using the feature of Cartesian product, we have easily analyzed the local stability of the steady state. The analytical results were numerically confirmed.

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