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Interval Estimation of Coupling Delay Time from Phase Time Series

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Abstract – We consider interval estimation of a time delay in coupling between oscillatory systems from observed time series. It is shown that asymptotic estimates, based on an empiric model in the form of first-order phase oscillators and maximum likelihood formalism, can lead to false conclusions about the value of the delay in two cases: (i) for nonlinear low-dimensional systems whose phases are well-defined but considerable amplitude fluctuations make phase description of the dynamics insufficient, (ii) for systems whose phases are not well-defined due to large amplitude fluctuations. We suggest an empirical criterion for diagnosis of such problematic situations and develop a modified estimator assuring low probability of false conclusions in those situations. Efficiency of the suggested estimator is demonstrated for benchmark systems with different dynamical properties, including stochastic and deterministically chaotic oscillators. An application of the approach to an analysis of large-scale climate processes is presented.

1. Introduction

Phase approximation is widely used to describe dynamics of oscillatory systems in nonlinear dynamics and oscillation theory [1,2]. Due to reduction of model dimensionality and retaining essential dynamical properties, phase description appears an efficient approach to a series of problems including the study of synchronization conditions [2,3]. Due to high sensitivity of the phase variable to external influences, phase dynamics analysis is used to detect couplings between oscillators from time series [4-6] which appears useful in different fields including neurophysiology and climatology (e.g. [7] and references therein). Within such an analysis, it is important to have tools for coupling delay estimation [8], e.g. to estimate signal propagation time in a complex medium. When dealing with short time series (several dozens of basic periods) typical in practice, it is important to get not only the value of the delay (a point estimate), but also a justified estimate of its uncertainty (an interval estimate). Several recent studies is devoted to the development of such tools [9,10]: the estimation methods are based on fitting empirical models in the form of coupled first-order phase oscillators where future phases are determined by the current phase values and external noises generated *independently* of the current phase values. Such a model is strictly justified for self-sustained oscillators which individually exhibit limit cycles, while couplings and noises slightly perturb these cycles [1]. Under violation of those conditions, including intensive noises leading to strong amplitude fluctuations or chaotic regimes of low-dimensional nonlinear systems, the phase description is insufficient and the first-order phase oscillators appear quite a rough approximation. Still, one may expect that the phase model-based coupling delay estimators are sometimes applicable even in such complicated cases, but it is to be checked. Further, for practical applications one needs a criterion to recognize situations where the existing delay estimators are erroneous. Then, one also needs modified estimators applicable to such problematic situations. The questions of revealing problematic situations, finding a criterion for their recognition, and developing a modified estimator are studied in this work. Section 2 describes an existing interval estimator of coupling delay. Section 3 presents a typical situation, where that interval estimator can be erroneous, and introduces a modified estimator assuring low pre-defined error probability. Section 4 gives an application of the approach to real-world data about large-scale climate processes El-Nino/Southern Oscillation and North Atlantic Oscillation. Conclusions are given in Section 5.

2. Asymptotic Interval Estimator

According to the technique developed in Refs. [8-10], one computes phases of the observed signals $x_1(t)$ and $x_2(t)$ (e.g. using the analytic signal construction [2] as in the examples below) and gets the time series $\{\varphi_1(t_1),\ldots,\varphi_1(t_N)\}$ and $\{\varphi_2(t_1),\ldots,\varphi_2(t_N)\}$ from observed signals $\{x_1(t_1),\ldots,x_1(t_N)\}$ and $\{x_2(t_1),\ldots,x_2(t_N)\}$**,** where $t_i = i\Delta t$, Δt is sampling interval, *N* is time series length. Then, one fits the phase dynamics model whose form comes from the fact that the phase dynamics of weakly perturbed periodic self-sustained oscillators yields to the differential equations of the first-order stochastic phase oscillators

$$d\varphi_k(t)/dt = \omega_k + G_k(\varphi_k(t), \varphi_j(t - \Delta^*_{j\to k})) + \xi_k(t), \quad k, j = 1,2,\ j \neq k, \tag{1}$$

where independent white noises possess covariance functions $\langle \xi_k(t)\xi_k(t')\rangle = D_{\xi_k}\delta(t-t')$, functions G_k determine both individual phase nonlinearity of the oscillators and their couplings, and $\Delta^*_{j\to k}$ are coupling delay times. In time series analysis, on fits stochastic difference equations which correspond to the equations (1) integrated over an interval of the finite length τ (a parameter of the method):

$$\varphi_k(t+\tau)-\varphi_k(t) = F_k(\varphi_k(t),\varphi_j(t-\Delta_{j\to k})) + \varepsilon_k(t),\quad k,j=1,2,\ j\neq k \tag{2}$$

where $\Delta_{j\to k}$ is a trial time delay, F_k is a low-order trigonometric polynomial, whose coefficients are determined via minimization of the mean squared model error

$$s_k^2 = \left\langle \left(\varphi_k(t_i+\tau)-\varphi_k(t_i) - F_k(\varphi_k(t_i),\varphi_j(t_i-\Delta_{j\to k}))\right)^2 \right\rangle_i .$$

An achieved minimal value $s_k^2(\Delta_{j\to k})$ is then minimized over $\Delta_{j\to k}$: its minimum point is $\hat{\Delta}_{j\to k} = \arg\min_{\Delta_{j\to k}} s_k^2(\Delta_{j\to k})$. An unbiased coupling delay estimate for the system (1) then reads $\hat{\Delta}^{corr}_{j\to k} = \hat{\Delta}_{j\to k} + \tau/2$. An asymptotic maximum likelihood estimator of its variance is given by

$$\hat{\sigma}^2_{\hat{\Delta}_{j\to k}} = \frac{2\hat{\sigma}^2_{\varepsilon_k}}{N'}\left(\left.\frac{\partial^2 s_k^2(\Delta_{j\to k})}{\partial \Delta^2_{j\to k}}\right|_{\Delta_{j\to k}=\hat{\Delta}_{j\to k}}\right)^{-1},$$

where N' is the number of statistically independent values of model residual errors over the time series. It is estimated as $N' = N\Delta t/L$, where $L=\max[T,\tau]$ and T is the decay time of the autocorrelation function of the model residual errors for the *k*th oscillator. The 95% confidence interval for the time delay is then given by $[\hat{\Delta}^{corr}_{j\to k} - 2\hat{\sigma}_{\hat{\Delta}_{j\to k}}; \hat{\Delta}^{corr}_{j\to k} + 2\hat{\sigma}_{\hat{\Delta}_{j\to k}}]$ and its width is $M = 4\hat{\sigma}_{\hat{\Delta}_{j\to k}}$. Efficiency of this interval estimator is shown in Refs. [9,10] for coupled phase oscillators perturbed by white or colored noise and for van der Pol generators. Erroneous estimation results are observed if the dependency $s_k^2(\Delta_{j\to k})$ does not exhibit a single clear minimum which occurs in case of too small noise level in the driving oscillator or too large noise level in the driven oscillator.

3. Problematic Situations and Modified Estimator

Let us consider low-dimensional nonlinear systems with well-defined phases and possibility of chaotic dynamics where problems with the above estimation technique can be expected due to violation of the assumption about one-dimensional phase dynamics perturbed by external noises (1). Such an example is given by Roessler systems:

$$\begin{aligned}
\dot{x}_1(t) &= -\omega_1 y_1(t) - z_1(t) + \xi_1,\\
\dot{y}_1(t) &= \omega_1 x_1(t) + a y_1(t),\\
\dot{z}_1(t) &= b - z_1(t)(r - x_1(t)),\\
\dot{x}_2(t) &= -\omega_2 y_2(t) - z_2(t) + K(x_1(t-\Delta) - x_2(t)) + \xi_2,\\
\dot{y}_2(t) &= \omega_2 x_2(t) + a y_2(t),\\
\dot{z}_2(t) &= b - z_2(t)(r - x_2(t)),
\end{aligned} \tag{3}$$

where $\omega_1 = 1.015, \omega_2 = 0.985$, $a=0.1$, $b=0.1$, and parameter r has been varied in a wide range allowing transition from periodic to chaotic regimes via a period-doubling cascade, $\xi_{1,2}$ are white noises of intensities $D_{\xi_{1,2}}$, coupling delay time is $\Delta_0 = 12$, K is coupling coefficient. If one defines phases via the relationships $x_{1,2} = A_{1,2}\cos\varphi_{1,2}$ and $y_{1,2} = A_{1,2}\sin\varphi_{1,2}$, then the phase dynamics of the driven system yields to the equation

$$\begin{aligned}
\frac{d\varphi_2}{dt} &= \omega_2 + \frac{z_2(t)\sin\varphi_2(t)}{A_2(t)} + \frac{(K+a)\sin 2\varphi_2(t)}{2} - \\
&\frac{KA_1(t)\sin(\varphi_2(t)-\varphi_1(t-\Delta))}{2A_2(t)} - \frac{\xi_2(t)\sin\varphi_2}{A_2(t)} - \\
&\frac{KA_1(t)\sin(\varphi_2(t)+\varphi_1(t-\Delta))}{2A_2(t)}.
\end{aligned} \tag{4}$$

Even at zero $D_{\xi_{1,2}}$ in the system (3), the reduced phase model (1) must include "noise" terms approximating the influence of the amplitude *A* and the third coordinate *z*. Properties of such "efficient phase noises" may be rather non-trivial, especially in chaotic regimes, leading to larger-than-expected errors in the above asymptotic estimates.

In numerical simulations we generated ensembles of 100 time series of the variables $x_{1,2}$ at each set of parameter values: the integration step in the Euler integration scheme was 0.001, the sampling interval Δt = 0.3 (20 data point per a basic period), the length of each time series N = 2000 (about 100 basic periods) or N = 20000 (i.e. 1000 periods), the parameter $\tau = 1.5$ (it gives an optimal sensitivity of the method, though the results are weakly sensitive to its values in the range from a quarter to several basic periods). From each pair of time series we computed their phases via Hilbert transform and obtained the above interval estimates of the delay. Then, the number of erroneous estimates (i.e. such that Δ does not belong to the interval $[\hat{\Delta}^{corr}_{j\to k} - 2\hat{\sigma}_{\hat{\Delta}_{j\to k}}; \hat{\Delta}^{corr}_{j\to k} + 2\hat{\sigma}_{\hat{\Delta}_{j\to k}}]$) was counted and the frequency of the errors f_{err} was calculated. We say that the estimator works properly if $f_{err} < 0.1$, because error probability of 0.05 corresponds to the claimed 0.95 confidence band and finite-ensemble fluctuations of the error frequency distributed according to Bernoulli's law make an allowable error frequency level somewhat larger. The results of the analysis for the individually chaotic oscillators with r = 10 and coupling coefficient $K = 0.05$ are shown in Fig.1. Autocorrelation function of the phase model residual errors (Fig.1,a) decays down to a small value (about 0.2) after several oscillations with doubled basic oscillation period due to peculiarities of the Roesller attractor structure. At time series length of 100 basic periods, situations "good" for the asymptotic interval estimator correspond to noise-perturbed regimes with $\sqrt{D_{\xi_1}} > 0.6$ (Fig.1,b, circles). At weaker noises, f_{err} exceeds the threshold level of 0.1, i.e. the asymptotic estimator becomes unreliable. The difficulty can be diagnosed in practice [9,10] from the absence of a clear minimum on the plot $s_2^2(\Delta_{1\to 2})$ (left inset in Fig.1,b).

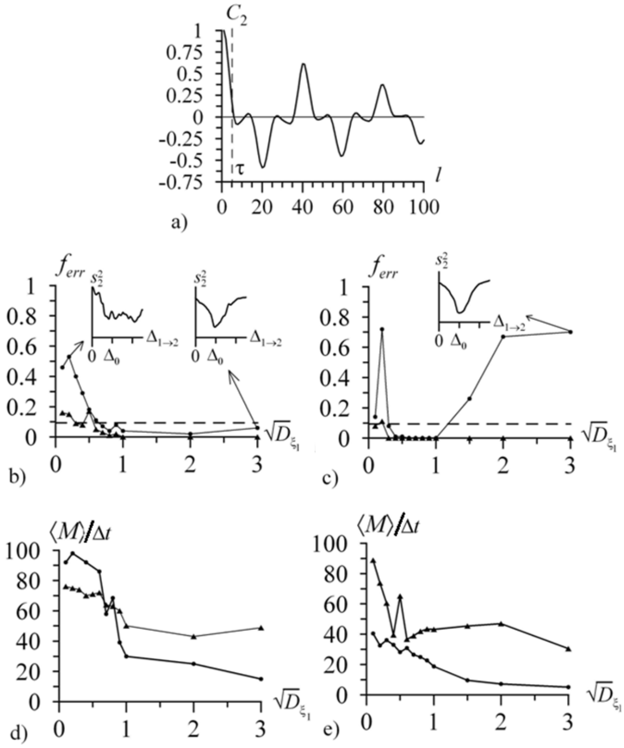

FIGURE 1. Coupling analysis for the system (3): Autocorrelation function of the phase model residual errors (a); error frequency for the asymptotic (circles) and modified (triangles) interval estimators at N = 2000 (b) and N = 20000 (c); the width M in units of Δt for the asymptotic (circles) and modified (triangles) interval estimators at N = 2000 (d) and N = 20000 (e).

The situation becomes even more difficult if longer time series are considered (1000 basic periods, probably less important in practical applications): large error frequency is observed at high noise levels as well (Fig.1,c, circles). It occurs because the point estimator $\hat{\Delta}_{j\to k}^{corr}$ appears somewhat biased, while the variance estimator $\hat{\sigma}^2_{\hat{\Delta}_{j\to k}}$ remains correct and becomes small (of the order of nonzero estimator bias) for long time series leading to frequent erroneous estimates. The results are analogous for other forms of couplings (e.g. couplings introduced into the equation for the y-variable) and for strongly perturbed individually-periodic systems. The cause of the nonzero bias seems to be in the peculiarities of the interaction between the phases and amplitudes and z-coordinates ignored in the phase model (2). The bias can be different for different nonlinear systems, so that one needs a special criterion for the practical recognition of possible difficulties and a modified estimator to get reliable estimates in such problematic situations.

We suggest to regard an essentially non-quadratic form of the minimum of $s_k^2(\Delta_{j\to k})$ as a sign of "danger". Namely, one should be careful in using the asymptotic interval estimator if the minimum is skewed (asymmetric, as in the right inset in Fig.1,b and inset in Fig.1,c), sufficiently deep local minima exist in the vicinity of the global minimum, etc. In such cases, we suggest to go beyond the asymptotic estimators based on approximation of local properties of the s_k^2 minimum and focus on its global properties which may be expected to be a more robust feature. Namely, we draw a straight line parallel to the abscissa axis at the mean level between maximal and minimal values of s_k^2 within the range of trial delays considered (e.g. a range from zero to five basic periods used above) and take its leftmost and rightmost cross-section points with the plot $s_k^2(\Delta_{j\to k})$ as the boundaries of the interval estimator. Such a modified (rough) estimator eliminates high error frequencies in all the above problematic situations (Fig.1,b,c, triangles) at the expense of typically somewhat wider confidence band M (Fig.1,d,e, triangles). However, it still allows an informative estimation, distinguishing an existing coupling delay from zero in the above examples. The modified estimator may also appear unreliable only if any clearly pronounced minimum on the plots $s_k^2(\Delta_{j\to k})$ (even a skewed one) is absent (left inset in Fig.1,b) which is easily diagnosed in practice.

The asymptotic estimator encounters the same large-error problems also in cases when the oscillators' phases are not well-defined due to strong amplitude fluctuations induced either by external random perturbations or internal chaotic dynamics. We showed that in numerical simulations with chaotic Lorenz systems and stochastic linear oscillators. The modified estimator allowed us to avoid frequent erroneous conclusions in all these cases as well.

4. Application to climate processes

Both estimators are applied to an analysis of couplings between El-Nino/Southern Oscillation (ENSO) and North Atlantic Oscillation (NAO) from observational data. These climate processes represent leading modes of interannual climate variability [11]. Influence of ENSO on

NAO was detected in Ref. [12] where a point estimate of the delay time in this influence was obtained. The latter appeared equal to about 20-24 months. The above interval estimators may give additional information about probable range of values of that delay. We have used the following indices reflecting those processes: the leading mode of 500 hPa geopotential height surface for NAO [13] and sea surface temperature in equatorial Pacific (region Nino-3.4: 5^{o}N-5^{o}S, 170^{o}W-120^{o}W) for ENSO [11]. The data under analysis cover the interval since 1950 till now and are available at http://www.ncep.noaa.gov.

As shown in Ref. [12], the phases defined via complex Morlet wavelet transform [14-16] in the frequency band corresponding to the periods from 24 to 40 months allow us to detect the influence of ENSO to NAO. Here, we applied the above interval estimators to the same phase time series. The results are following: The corrected point estimate of the delay time is 36 months. The asymptotic interval estimate cover the range 29-43 months. However, the plot for the model prediction errors reveals sings that the modified estimator may be more appropriate. The latter gives a wider confidence interval from 8 to 47 months. Still, not approaches detect nonzero coupling delay. However, the modified estimators evidences that it is not necessary to seek for the causes of the large 2-year delay. Quite probably, the delay is smaller and consists slightly more than half a year. Such information is of interest for climate science as discussed in Ref. [12].

5. Conclusions

We have studied applicability of the asymptotic interval estimator of the coupling delay time based on phase dynamics modeling, to oscillators with different dynamical properties. We have shown that low-dimensional nonlinear dynamics along with strong amplitude fluctuations can strongly increase probability of erroneous conclusions about the value of the delay. An empirical criterion for diagnosis of such problematic situations is suggested, along with a modified interval estimator based on the rough (global or larger-scale) properties of the phase model residual errors' minimum. The latter estimator is shown to provide the error probability less than a pre-defined small value for characteristic oscillatory systems with rather different properties of phase dynamics. Hence, it extends possibilities of reliable coupling delay time estimation for a wide range of oscillatory systems in practice. In particular, we have applied the suggested approach to analyze large-scale climate processes from observational data where it confirms a non-zero delay in the influence of El-Nino/Southerm Oscillation on the North Atlantic Oscillations.

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