2016 International Symposium on Nonlinear Theory and Its Applications,
NOLTA2016, Yugawara, Japan, November 27th-30th, 2016

Steady State Analysis of Digital Return Maps and Cellular Automata

Naoki Tada†, Hiroki Yamaoka‡ and Toshimichi Saito†

†EE Dept., Koganei, Tokyo 184-8584, Japan, Email: tsaito@hosei.ac.jp
‡Hitachi Kokusai Electric Inc., Chiyoda-ku, Tokyo 101, Japan

Abstract—This paper considers dynamic of digital return maps, simple digital dynamical systems on a set of points. Depending on parameters and initial states, the map can generate various periodic orbits. In order to analyze the steady state, we present two feature quantities: periodic orbits occupancy rate and dispersion of periods of the orbit. Using the feature quantities, we construct the feature quantities plane that is useful in visualization and classification of the dynamics. Using the feature quantities, a simple class of cellular automata are analyzed.

1. Introduction

A digital return map (Dmap [1] [2]) is a simple digital dynamical system on a set of points. It can be regarded as a digital version of analog one-dimensional maps represented by the logistic map [3] [4]. Since the number of the points is finite, the Dmap cannot generate chaos. However, depending on parameters and initial states, the Dmap can generate a variety of periodic orbits (PEO) in the steady state.

The Dmap is related to various digital dynamical systems and their applications including cellular automata and signal/image processing [5]-[8], and dynamic binary neural networks and control of switching circuits [9]. Analysis of the Dmap can contribute not only to basic study of nonlinear dynamics but also to engineering applications. However, the analysis is hard because the Dmap has a large variety and the dynamics is very complicated.

In order to analyze steady states of Dmaps, this paper presents two simple feature quantities. The first quantity is occupancy rate of the PEOs for all the initial points. It can characterize plentifulness of the steady states. The second quantity is dispersion of periods of the PEOs. It can characterize variety of the PEOs. Using the two feature quantities, a feature plane is constructed. The feature plane is useful in visualization and classification of the dynamics.

As an example of the Dmaps, we consider a Dmap derived from a simple class of elementary cellular automata (ECAs) on 8 dimensional binary spaces. The dynamics of ECA is governed by one rule and can generate a variety of spatiotemporal patterns, The dynamics of the ECAs is integrated into a Dmap on a set of 2^8 points. Calculated the feature quantities of the ECA, we visualize/classify rich dynamics of the ECA. Note that this is the first publication of the feature plane for steady state analysis and its application to ECAs.

2. Digital return map and simple feature quantities

The Dmap is defined on a set of points to itself and its iteration generates a digital sequence as shown in Fig. 1:

$$\begin{aligned} &\theta_{n+1} = f(\theta_n),\ \theta_n \in L_N \\ &L_N \equiv \{l_1, \cdots, l_N\},\ l_i \equiv i/N,\ i = 1 \sim N \end{aligned} \tag{1}$$

where θ_n is a digital state variable on L_N at discrete time n and L_N is a set of N points l_i. Since the points are equivalent to binary vectors, we refer to this systems as to be digital.

Since the number of points in the domain L_N is finite, steady state must be a periodic orbit. Here we give basic definitions.

Definition 1: A point $\theta_p \in L_N$ is said to be a periodic point with period p if $f^p(\theta_p) = \theta_p$ and $f^q(\theta_p) \neq \theta_p$ for $0 < q < p$ where f^p is the p-fold composition of f. A PEO with period 1 is referred to as a fixed point. A sequence of the periodic points $\{f(\theta_p), \cdots, f^p(\theta_p)\}$ is said to be a periodic orbit (PEO) with period p.

Definition 2: A point θ_e is said to be an eventually periodic point (EPP) if θ_e is not a periodic point and there exists some positive integer k such that $f^k(\theta_e)$ is a periodic point.

The Dmap for $N = 16$ in Fig. 1 has one fixed point and one PEO with period 3. The other 12 points are EPPs.

In order to consider the steady state dynamics, we introduce two simple feature quantities. First, let N_p be the number of periodic points on N points. The first quantity is

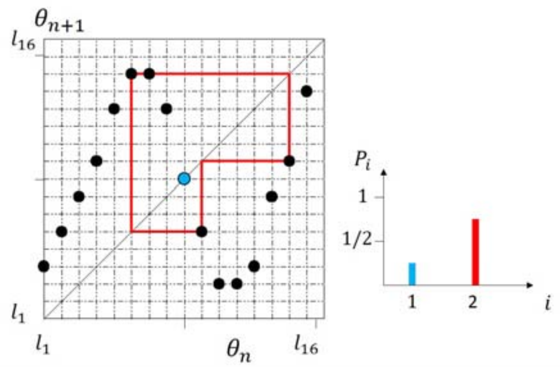

Figure 1: A digital return map (Dmap) and distribution of periods. The blue point is a fix point and $P_1 = 1/4$. The red orbit is a periodic obit (PEO) with period 3 and $P_2 = 3/4$. Black points are eventually periodic points (EPPs).

the PEOs occupancy rate over the domain of the Dmap

$$\alpha = \frac{N_p}{N}, \ \frac{1}{N} \leq \alpha \leq 1 \tag{2}$$

This quantity characterize plentifulness of steady states.

In order to define the second feature quantity, we define several notations. Let a Dmap have N_p pieces of periodic points. Let N_e be the number of PEOs and let p_i be the period of the i-th PEO, $i = 1 \sim N_e$. Let $P_i = p_i/N_P$ where $\sum_{i=1}^{N_e} P_i = 1$. $\{P_i\}$ is referred to as a period distribution. The second quantity γ is dispersion of periods of the PEOs.

$$\gamma = \sum_{i=1}^{N_e} P_i^2, \ \frac{1}{N} \leq \gamma \leq 1 \tag{3}$$

It can characterize variety of PEOs. If a Dmap has one PEO then γ takes the maximum value $\gamma = 1$. If a Dmap has N fixed points then γ takes the minimum value $\gamma = 1/N$. In the Dmap in Fig. 1, one fixed point and one PEO with period 3 correspond to $P_1 = 1/4$ and $P_2 = 3/4$, respectively. Hence this Dmap is characterized by $\gamma = 10/16$ and $\alpha = 4/16$.

In order to consider the dynamics, the α versus γ feature plane is constructed as shown in Fig. 2. On the α-γ feature plane, we define three criterial objects for consideration of the dynamics. The first criteria is defined for a Dmap in Fig. 2 (a): if a Dmap has only one PEO then (α, γ) is plotted on

$$\text{Unique line } l_u\text{: } \gamma = 1 \tag{4}$$

The second criterion is defined for a Dmap in Fig. 2 (b): if a Dmap has no EPP (no transient phenomenon) and periodic points are dense then (α, γ) is plotted on

$$\text{Dense line } l_d\text{: } \alpha = 1 \tag{5}$$

The third criterion is defined for a Dmap in Fig. 2 (d): if all the PEOs are fixed points then (α, γ) is plotted on

$$\text{Fixed point curve } C_f\text{: } \gamma = \frac{1}{N\alpha} \tag{6}$$

In the feature plane, we note three end points. If the Dmap has only one fixed point such as Fig. 2(a) then it is plotted at the left top point $(\alpha, \gamma) = (1/N, 1)$. If the Dmap has N fixed points and has no EPPs such as Fig. 2(b) then it is plotted at the right bottom point $(\alpha, \gamma) = (1, 1/N)$. If the Dmap has one PEO with period N and has no EPP such as an M-sequence in Fig. 2(c) then it is plotted at the right top corner $(\alpha, \gamma) = (1, 1)$.

3. Cellular automata and digital return maps

We consider ECAs on the ring of M cells. Let $x_i^t \in \{0, 1\} \equiv \boldsymbol{B}$ be the binary state of the i-th cell at discrete time t where $i = 1 \sim M$. The time evolution of x_i^t is governed by a Boolean function of x_i^t and its closest neighbors:

$$x_i^{t+1} = F_i(x_{i-1}^t, x_i^t, x_{i+1}^t), \ i = 1 \sim M \tag{7}$$

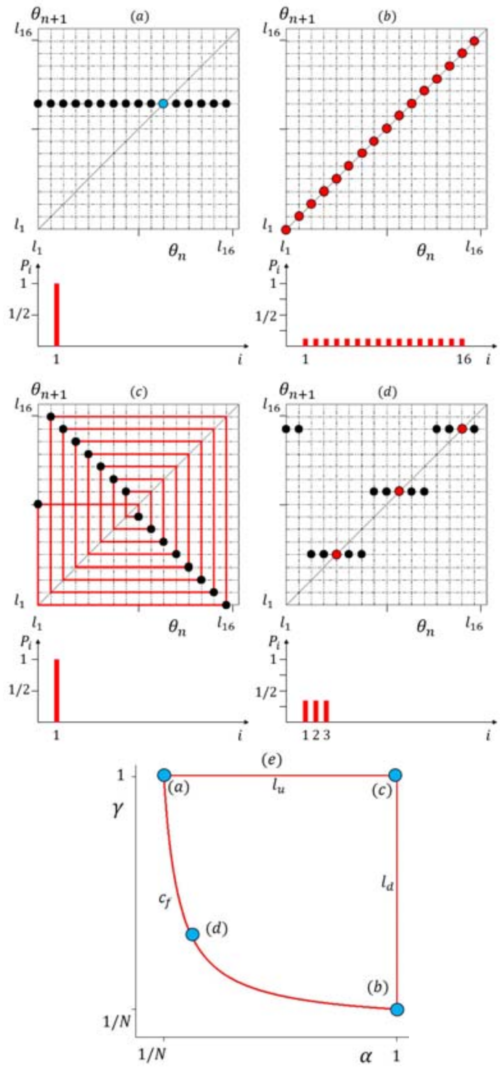

Figure 2: Typical Dmaps for $N = 16$. (a) One fixed point. $(\alpha, \gamma) = (1/16, 1)$. (b) 16 fixed points and no EPPs. $(\alpha, \gamma) = (1, 1/16)$. (c) One PEO with period 16 (M-sequence) and no EPP. $(\alpha, \gamma) = (1, 1)$. (d) Three fixed points. $(\alpha, \gamma) = (3/16, 1/3)$. (e) $\alpha - \gamma$ feature plane. l_u: Unique line. l_d: Dense line. c_f: Fixed point curve.

where $x_{-1}^t \equiv x_M^t$ and $x_{i+1}^t \equiv x_1^t$ on the ring. F_i is referred to as a rule. In the ECA, F_i does not depend on i ($F_i = F$) and the dynamics is defined by one rule

$$\begin{array}{lll} y_0 = F(0,0,0), & y_1 = F(0,0,1), & y_2 = F(0,1,0) \\ y_3 = F(0,1,1), & y_4 = F(1,0,0), & y_5 = F(1,0,1) \\ y_6 = F(1,1,0), & y_7 = F(1,1,1). & \end{array} \tag{8}$$

where $y_j \in \boldsymbol{B}$ and $j = 0 \sim 7$. $(y_0, \cdots, y_7)$ is referred to as a rule table and is equivalent to an 8 bits binary number that is referred to as the rule number (RN) [5]. There exist 2^{2^3} rules for the ECA.

Fig. 3 shows some of spatiotemporal patterns from ECAs. Fig. 3(a) shows periodic pattern in the steady state with period 8. Fig. 3(b) shows a transient pattern falls into

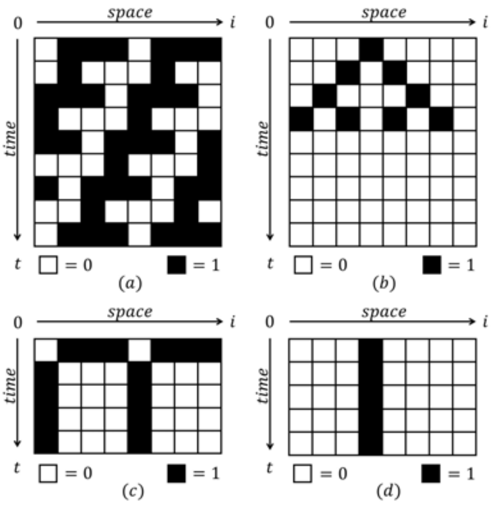

Figure 3: Examples of spatiotemporal patterns. (a) RN30 (b) RN90 (c) RN36 (d) RN102

the steady state of all 0. Fig. 3(c) and (d) shows fixed patterns in steady states. In order to analyze the dynamics, we introduce the Dmap. That is, an ECA of M cells is equivalent to the mapping from B^M to itself:

$$\boldsymbol{x}^{t+1} = F_D(\boldsymbol{x}^t),\ \boldsymbol{x}^t \equiv (x_1^t, \cdots, x_M^t) \in \boldsymbol{B}^M \tag{9}$$

Since $\boldsymbol{B}^M$ is equivalent to a set of 2^M points $I_D \equiv \{C_1, \cdots, C_{2^M}\}$, F_D is equivalent to the Dmap from I_D to itself.

We focus on the case $M = 8$ ($N = 2^M = 256$) in this paper. Fig. 4(a) shows a Dmap and its PEO with peirod 8 corresponding to the spatiotemporal pattern in Fig. 3(a). The Dmap has 5 PEOs as shown in the histogram. This Dmap is characterized by $(\alpha, \gamma) = (0.2, 0.64)$ and plotted at cross A in the feature plane in Fig. 5. The cross A corresponds to the following 4 rules.

$$\text{RS-A} = \{30, 86, 135, 149\} \tag{10}$$

Fig. 4(b) shows a Dmap corresponding to the spatiotemporal pattern in Fig. 3(b). The steady state is one fixed point only. This Dmap is characterized by $(\alpha, \gamma) = (0.004, 1)$ and plotted at cross B in the feature plane in Fig. 5. The cross B corresponds to the following 12 rules.

$$\begin{aligned}\text{RS-B} = \{&0, 8, 60, 64, 90, 102, 153,\\ &165, 195, 239, 253, 255\}\end{aligned} \tag{11}$$

Fig. 4(c) shows a Dmap corresponding to the spatiotemporal pattern in Fig. 3(c). This Dmap has 21 fixed points. This Dmap is characterized by $(\alpha, \gamma) = (0.08, 0.05)$ and plotted at cross C in the feature plane in Fig. 5. The cross C corresponds to the following 8 rules.

$$\text{RS-C} = \{36, 44, 72, 100, 203, 217, 219, 237\} \tag{12}$$

The spatiotemporal pattern in Fig. 3(d) corresponds to a Dmap having 256 fixed points

$$\boldsymbol{x} = F(\boldsymbol{x}),\ \boldsymbol{x} \in \boldsymbol{B}^M \tag{13}$$

This Dmap is characterized by $(\alpha, \gamma) = (1, 0.004)$ and plotted at cross D in the feature plane in Fig. 4 (e). The cross D corresponds to only RN204.

In the feature plane, red points are plots of all the ECAs. It suggests that the ECA can exhibit various dynamics even in such a small case of $M = 8$.

4. Conclusions

In order to analyze steady states of Dmaps, two simple feature quantities and feature plane are presented in this paper. Using the feature quantities, a simple class of ECAs are analyzed. Future problems include more detailed analysis of ECAs, analysis of mixed rule cellular automata[10], and engineering applications such as information compressions[6].

References

[1] N. Horimoto and T. Saito, Analysis of digital spike maps based on bifurcating neurons, NOLTA, IEICE, 3, 4, pp. 596-605, 2012.

[2] H. Yamaoka, N. Horimoto, and T. Saito, Basic feature quantities of digital spike maps. (S. Wermter et al. (Eds.): ICANN 2014), LNCS 8681, pp. 73-80, Springer, Sept. 2014.

[3] E. Ott, Chaos in dynamical systems, Cambridge, 1993.

[4] A. Lasota and M. C. Mackey, Chaos, fractals, and noise. Springer-Verlag, 1994

[5] L. O. Chua, A nonlinear dynamics perspective of Wolfram's new kind of science, I, II. World Scientific, 2005.

[6] W. Wada, J. Kuroiwa, and S. Nara, Completely reproducible description of digital sound data with cellular automata, Physics Letters A 306, pp. 110-115, 2002.

[7] J. D. Lohn, J. A. Reggia: Automatic discovery of self-replicating structures in cellular automata, IEEE Trans. Evolutionary Computation, 1, 3, 165–178 (1997)

[8] P. L. Rosin, Training cellular automata for image processing, IEEE Trans. Image Process., 15, 7, pp. 2076-2087, 2006.

[9] R. Kouzuki and T. Saito, Learning of simple dynamic binary neural networks, IEICE Trans. Fundamentals, E96-A, 8, pp. 1775-1782, 2013.

[10] N. Tada and T. Saito, Analysis of mixed-rule cellular automata based on simple feature quantities, S. Arik et al. (Eds.): ICONIP 2015, Part II, LNCS 9490, pp. 484-491, 2015.

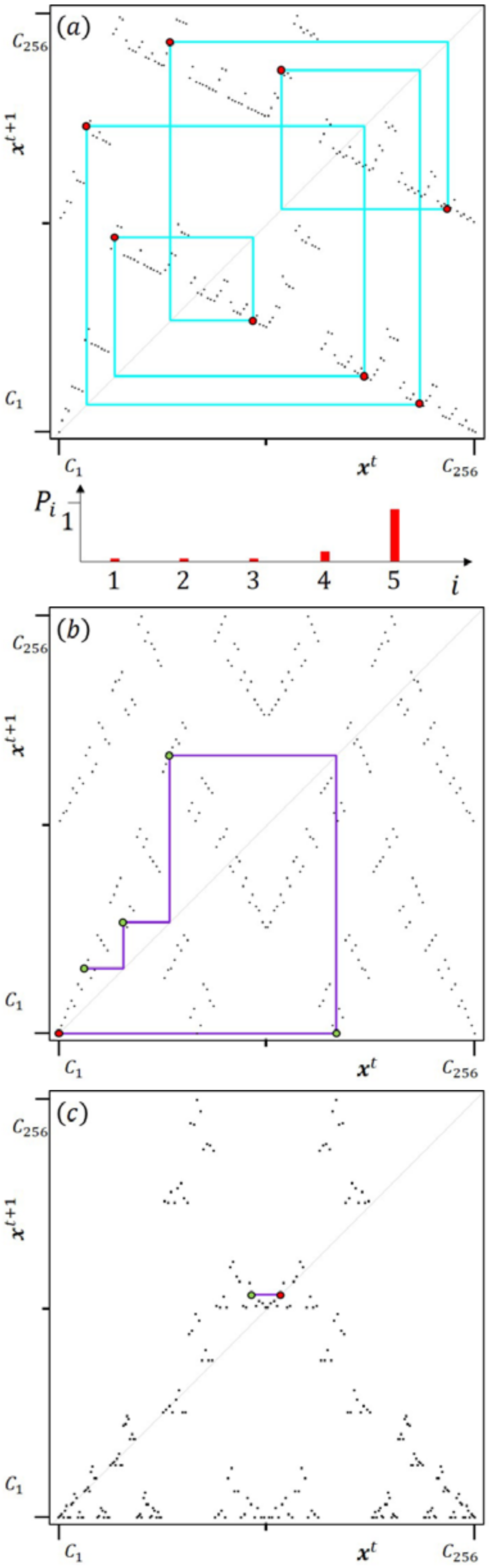

Figure 4: Examples of Dmaps from ECAs. (a) RN30, $(\alpha, \gamma) = (0.2, 0.64)$. Red points denote PEP. (b) RN90, $(\alpha, \gamma) = (0.004, 1)$. (c) RN36, $(\alpha, \gamma) = (0.08, 0.05)$.

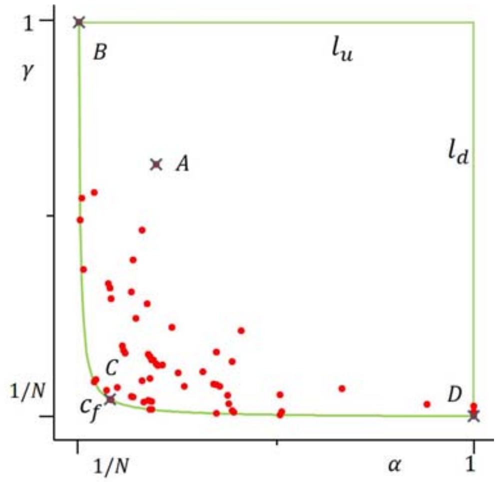

Figure 5: $\alpha - \gamma$ feature plane. l_u: Unique line. l_d: Dense line. c_f: Fixed point curve.