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# An Effective Stability Analysis Method for the Linear Impact Oscillators

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**Abstract**—In this study, we propose a stability analysis method for the linear impact oscillators. First, we show a two-dimensional linear impact oscillator. Next, we define the periodic solution and explain its stability. Then, we propose a stability analysis method using the Filippov's theory. Finally, we apply the method to a rigid overhead wire-pantograph system for confirming validity of the proposed method.

## 1. Introduction

The impact oscillation is observed in various fields of the engineering. In the mechanical field, for example, most of the component impacts with the other component [1, 2]. In the ecological and biological field, the simplified models, such as the forest fire model and the spiking neuron model, are proposed using the impact characteristics for rigorously understanding property of them [3, 4]. Generally, these systems are categorized as the impact oscillator. Since the impact oscillator is used in various field of engineering, it is important to clarify the fundamental characteristics of them.

It is known that the bifurcation phenomena are observed in the impact oscillators upon varying the parameters such as the coefficient of restitution between the mass points and frequency and amplitude of the external force [5]. Because of the dynamic behavior of the system qualitatively changes via the bifurcation phenomena, it is important to determine the bifurcation parameter for designing the optimal parameter of the system. Generally, for determining the bifurcation parameter of the impact oscillators, we have to calculate stability of the orbit. Some stability analysis methods are proposed: the Poincaré map approach [5, 6] is well-known stability analysis method in the field. In addition, we should not forget that the Yoshitake's algorithm [7], because it might be a first stability analysis method for the impact oscillators. In theory, we can calculate stability of the periodic orbit of almost all of the impact oscillators using these stability analysis methods. But, it is not easy for most of the engineers, who are not specialist of the bifurcation analysis, to calculate stability of the orbit due to the complicated algorithm and its technical implementation process. For solving these problems, we focus on a stability analysis method using the Filippov's theory [8]. The advantage of the method is its simplicity. We do not have to derive the Poincaré map and its derivative. The motion equations, the switching condition and the information of the orbit at the switching surface are only needed to the method. But, the Filippov's method is only available for the impact oscillators with the fixed border [9]. There is many impact oscillators with the moving border [1, 2, 10], and therefore, it is valuable to propose the stability analysis method for them using the Filippov's theory. As the first step to construct the stability analysis method, using the Filippov's method, for the $n$-dimensional impact oscillators with the moving border, we have proposed the method for the two-dimensional impact oscillator with the moving border [11]. However, the method is only available for the fixed point.

In this study, we improve the method for calculating the stability of the periodic orbit of the two-dimensional impact oscillators with the moving border. First, we show the behavior of the waveforms of the two-dimensional impact oscillator with the moving border. Note that we consider the periodic border for the simplicity. Then, we explain the Floquet theory [12] for understanding stability of the periodic orbit. Next, we discuss the *monodromy matrix*, which is a key of the method. Finally, we apply the proposed method to the rigid overhead wire-pantograph system. We will confirm the validity of the proposed method and its advantage.

## 2. The proposed method

### 2.1. The two-dimensional impact oscillator

The motion equation is given by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}, \tag{1}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are the 2 by 2 subsystem matrix and $\boldsymbol{x} = (x, v)^\top$. $x$ denotes the displacement and $v$ denotes the velocity of the mass. Let a moving (periodic) border $S(t)$ be

$$S(t) = a \sin \omega t. \tag{2}$$

When the impact phenomenon occurs between the mass and border, the velocity of the mass changes from $v_-$ to $v_+$ instantly.

We briefly explain the Floquet theory [12] for understanding stability of the orbit. Let the period-$m$ orbit as

$$\boldsymbol{x}(0) - \boldsymbol{x}(mT) = 0, \tag{3}$$

where $\boldsymbol{x}(t)$ is the solution of Eq. (1), and $T$ denotes the period of the border. Let the period-$m$ orbit, which satisfies Eq. (3), be $\boldsymbol{x}^*(t)$. The perturbation, between the period-$m$ orbit $\boldsymbol{x}^*(t)$ and its closed-orbit $\boldsymbol{x}(t)$ at $t = 0$, is expressed as:

$$\Delta\boldsymbol{x}_0 = \boldsymbol{x}^*(0) - \boldsymbol{x}(0). \tag{4}$$

Likewise, that of at $t = mT$ is

$$\Delta\boldsymbol{x}_m = \boldsymbol{x}^*(mT) - \boldsymbol{x}(mT). \tag{5}$$

Figure 1 shows an image of the period-$m$ orbit and its closed-orbit of the two-dimensional impact oscillator with the periodic border. The relationship between $\Delta\boldsymbol{x}_0$ and $\Delta\boldsymbol{x}_m$ is expressed as:

$$\Delta\boldsymbol{x}_m = \boldsymbol{M}\Delta\boldsymbol{x}_0, \tag{6}$$

where

$$\boldsymbol{M} = \boldsymbol{M}_0 \circ \boldsymbol{M}_1 \circ \ldots \circ \boldsymbol{M}_{m-1} \tag{7}$$

is a 2 by 2 matrix. The matrix $\boldsymbol{M}$ is called as the monodromy matrix and characteristic multipliers of the monodromy matrix $\mu_1$ and $\mu_2$ is called as the Floquet multipliers. If the Floquet multipliers satisfies $|\mu_1| < 1$ and $|\mu_1| < 1$, the period-$m$ orbit is stable and otherwise unstable. In the following analysis, we discuss the monodromy matrix, i.e., how to calculate $\boldsymbol{M} = \boldsymbol{M_0} \circ \boldsymbol{M_1} \circ \ldots \circ \boldsymbol{M}_{m-1}$. Note that the two-dimensional impact oscillator and stability of the orbit have already been explained in Ref. [11]. However, we are not able to explain the proposed theory without them, and therefore, we explain them again in this paper.

### 2.2. The monodromy matrix of the period-$m$ orbit

We consider the perturbation of the orbit shown in Fig. 1. The red orbit denotes the period-$m$ orbit. The mass impacts with the border $l$ times. An example of the impact phenomenon is shown in the right-hand side in Fig. 1. Let an impact point of the period-$m$ orbit as $\boldsymbol{x}^*(\hat{t})$. Since the velocity of the mass jumps from $v_-$ to $v_+$ via the impact, the orbit $\boldsymbol{x}^*(\hat{t})_-$ jumps to $\boldsymbol{x}^*(\hat{t})_+$. Likewise, the perturbed orbit $\boldsymbol{x}(t)$, whose initial value $\boldsymbol{x}(0)$ is close to $\boldsymbol{x}^*(0)$, reaches to the impact point at $t = \bar{t}$ and orbit jumps from $\boldsymbol{x}(\bar{t})_-$ to $\boldsymbol{x}(\bar{t})_+$. Here, the perturbations of the orbit $\Delta\boldsymbol{x}_-$ and $\Delta\boldsymbol{x}_+$, just before and after the impact phenomenon occurs, are expressed as

$$\Delta\boldsymbol{x}_- = \boldsymbol{x}^*(\hat{t})_- - \boldsymbol{x}(\bar{t})_-, \tag{8}$$

$$\Delta\boldsymbol{x}_+ = \boldsymbol{x}^*(\hat{t})_+ - \boldsymbol{x}(\bar{t})_+. \tag{9}$$

Thus, the perturbation via the impact phenomenon is defined as

$$\Delta\boldsymbol{x}_+ = \boldsymbol{S}\Delta\boldsymbol{x}_-, \tag{10}$$

where a 2 by 2 matrix $\boldsymbol{S}$ is called as the saltation matrix.

The monodromy matrix of the period-$m$ orbit is defined as Eq. (7). We derive $\boldsymbol{M}_n$ as an example, where $n = 1, 2, \ldots, m-1$. Here, we assume the impact occurs only once for the sake of the simplicity; of course we can derive $\boldsymbol{M}_n$ if the impact occurs $k$ times. Under this situation, a part of the monodromy matrix $\boldsymbol{M}_n$ is expressed as follows:

$$\boldsymbol{M}_n = e^{\boldsymbol{A}(T-\bar{t})}\boldsymbol{S}e^{\boldsymbol{A}\bar{t}}, \tag{11}$$

where we use Taylor expansion for calculating $e^{\boldsymbol{A}t}$. So, the stability of the period-$m$ orbit of the impact oscillators with the periodic border can calculate using

$$\det(\mu\boldsymbol{I}_2 - \boldsymbol{M}) = 0, \tag{12}$$

where $\boldsymbol{I}_2$ is the unit matrix and $\mu$ is the Floquet multipliers (characteristic multipliers). Note that we can calculate the stability of the system with fixed border using the same algorithm if we assume $a = 0$ in Eq. (2).

## 3. Stability analysis using the proposed method

We apply the method to a two-dimensional impact oscillator [10] shown in Fig. 2. The system composes a spring,

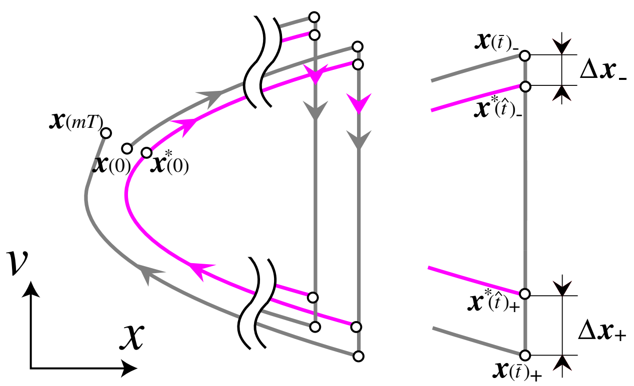


Figure 1: Example of the orbit.

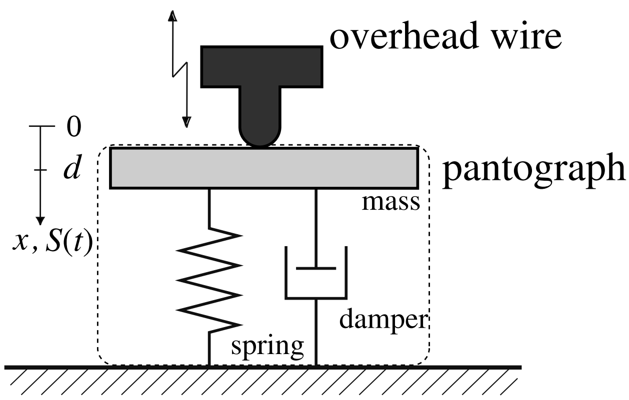


Figure 2: A two-dimensional impact oscillator.

damper and mass. The mass impacts with the oscillating object (border). The motion equation of the mass is given by

$$\begin{cases} \dfrac{dx}{dt} &= v \\ \dfrac{dv}{dt} &= -x - 2\zeta v \end{cases}, \tag{13}$$

where $\zeta$, $x$ and $v$ denote the damping ratio, displacement and velocity of the mass, respectively. The motion equation (displacement) of the border is expressed as

$$S(t) = \varepsilon \sin \omega t + 1, \tag{14}$$

where $\varepsilon$ and $\omega$ are the amplitude and the angular frequency.

Figure 3 shows behavior of the waveforms; $x(t)$ and $v(t)$ denote the displacement and velocity of the mass. If the mass impacts with the border, the velocity of the mass changes from $v_-$ to $v_+$ instantly. Here, the coefficient of restitution between the mass and border is expressed as

$$\alpha = \frac{v_+ - \dfrac{dS(t)}{dt}}{\dfrac{dS(t)}{dt} - v_-}. \tag{15}$$

Thus, the relationship between the velocities $v_-$ and $v_+$ are described as

$$v_+ = -\alpha v_- + (1 + \alpha)\frac{dS(t)}{dt}. \tag{16}$$

Figure 4 shows the 1-parameter bifurcation diagram upon varying the frequency of the border $\omega$ as the bifurcation parameter. Note that we sample the data at every period of $T = 2\pi/\omega$ and get the 1-parameter bifurcation diagram; an image of the sampled data is shown as the white points in Fig. 3. We can observe that the period-1 orbit bifurcates to the period-2 orbit around $\omega = 6.15$ and period-2 orbit bifurcates to the period-4 orbit around $\omega = 6.42$. In the figure, the bifurcation points are shown as $\mathrm{PD}^1$ and $\mathrm{PD}^2$ where index numbers mean the periodicity of the orbit concerning with the bifurcation phenomenon. In the following analysis, we calculate stability of the period-1 and period-2 orbit as the examples.

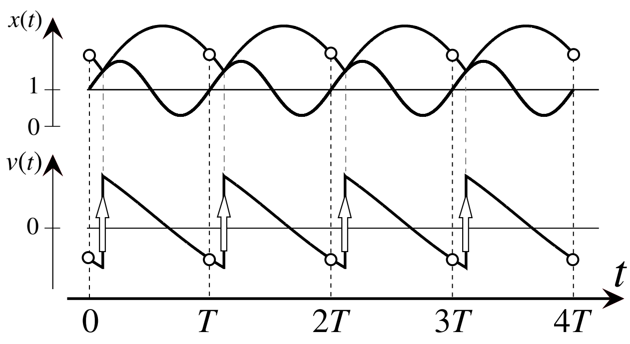


Figure 3: The behavior of the waveforms.

The monodromy matrix for the period-1 orbit is expressed as

$$\boldsymbol{M} = e^{\boldsymbol{A}\bar{t}_{\rm i}}\boldsymbol{S}e^{\boldsymbol{A}\bar{t}'_{\rm i}}. \tag{17}$$

Likewise, that for the period-2 orbit is expressed as

$$\boldsymbol{M} = \boldsymbol{M}_0 \circ \boldsymbol{M}_1, \tag{18}$$

where $\boldsymbol{M}_0 = e^{\boldsymbol{A}\bar{t}_{\rm i}}\boldsymbol{S}e^{\boldsymbol{A}\bar{t}'_{\rm i}}$ $\boldsymbol{M}_1 = e^{\boldsymbol{A}\bar{t}_{\rm i}}\boldsymbol{S}e^{\boldsymbol{A}\bar{t}'_{\rm i}}$. In Eqs. (17) and (18), the saltation matrix $\boldsymbol{S}$ is described as

$$\boldsymbol{S} = \begin{pmatrix} 1 - \dfrac{v_- - v_+}{v_- - \omega\epsilon\cos\omega\bar{t}_{\rm i}} & 0 \\ -\dfrac{g(\bar{t}_{\rm i})}{v_- - \omega\epsilon\cos\omega\bar{t}_{\rm i}} & -\alpha \end{pmatrix}, \tag{19}$$

where $g(\bar{t}_{\rm i})$ is

$$\begin{aligned} g(\bar{t}_{\rm i}) = \alpha\left(\bar{x}(\bar{t}_{\rm i}) + 2\gamma v_-\right) + \left(\bar{x}(\bar{t}_{\rm i}) + 2\gamma v_+\right) \\ - (1+\alpha)\omega^2\epsilon\sin\omega\bar{t}_{\rm i}. \end{aligned} \tag{20}$$

A time $\bar{t}_{\rm i}$ is a critical time when the mass, which starts at time $kT$, impacts with the border; $\bar{t}'_{\rm i}$ satisfies $\bar{t}'_{\rm i} = T - \bar{t}_{\rm i}$. In addition, the subsystem matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -1 & -2\gamma \end{pmatrix}. \tag{21}$$

The stability of the periodic orbits are calculated based on Eq. (12).

Tables 1 and 2 show the characteristic multipliers of the period-1 and period-2 orbit. Note that we calculate the characteristic multipliers using the Poincaré map approach [5, 6] for comparing with the proposed method (see Table 1). Table 2 says that the period doubling bifurcation occurs at $\omega = 6.1459$ and $\omega = 6.4188$. The bifurcation points seems to be correct in the 1-parameter bifurcation diagram (see Fig. 4). Moreover, the accuracy of calculation can be said good. So, we conclude that the proposed

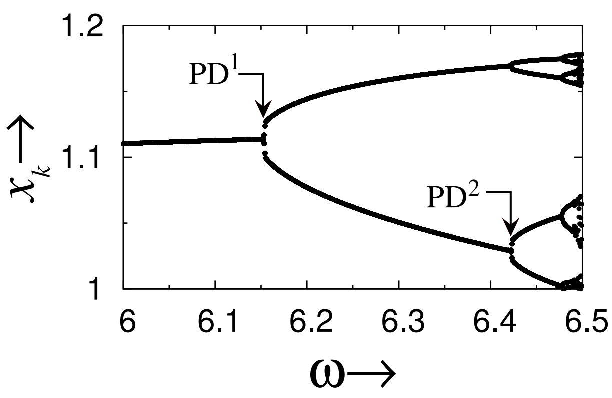


Figure 4: The 1-parameter bifurcation diagram ($\alpha = 0.5$, $\varepsilon = 0.068$, $\zeta = 0.4$).

Table 1: Analytical results of the Poincaré map approach ($\alpha = 0.5$, $\varepsilon = 0.068$, $\zeta = 0.4$).

| Period-1 | | | | Period-2 | | | |
|---|---|---|---|---|---|---|---|
| $\omega$ | $\mu_1$ | $\mu_2$ | Remarks | $\omega$ | $\mu_1$ | $\mu_2$ | Remarks |
| 6.1350 | -0.1123 | -0.9803 | Stable period-1 orbit | 6.4050 | -0.0145 | -0.8923 | Stable period-2 orbit |
| 6.1400 | -0.1114 | -0.9893 | Stable period-1 orbit | 6.4100 | -0.0139 | -0.9318 | Stable period-2 orbit |
| 6.1450 | -0.1105 | -0.9983 | Stable period-1 orbit | 6.4150 | -0.0132 | -0.9715 | Stable period-2 orbit |
| 6.1459 | -0.1103 | -1.0000 | $PD^1$ | 6.4185 | -0.0130 | -1.0000 | $PD^2$ |
| 6.1500 | -0.1096 | -1.0072 | Unstable period-1 orbit | 6.4200 | -0.0129 | -1.0112 | Unstable period-2 orbit |

Table 2: Analytical results of the proposed method ($\alpha = 0.5$, $\varepsilon = 0.068$, $\zeta = 0.4$).

| Period-1 | | | | Period-2 | | | |
|---|---|---|---|---|---|---|---|
| $\omega$ | $\mu_1$ | $\mu_2$ | Remarks | $\omega$ | $\mu_1$ | $\mu_2$ | Remarks |
| 6.1350 | -0.1124 | -0.9805 | Stable period-1 orbit | 6.4050 | -0.0146 | -0.8931 | Stable period-2 orbit |
| 6.1400 | -0.1114 | -0.9895 | Stable period-1 orbit | 6.4100 | -0.0140 | -0.9330 | Stable period-2 orbit |
| 6.1450 | -0.1105 | -0.9984 | Stable period-1 orbit | 6.4150 | -0.0134 | -0.9710 | Stable period-2 orbit |
| 6.1459 | -0.1103 | -1.0000 | $PD^1$ | 6.4188 | -0.0131 | -1.0000 | $PD^2$ |
| 6.1500 | -0.1096 | -1.0074 | Unstable period-1 orbit | 6.4200 | -0.0129 | -1.0104 | Unstable period-2 orbit |

method is able to calculate the stability of the periodic orbit of the two-dimensional impact oscillators with the periodic border. In addition, we should not forget that the propose method do not need the Poincaré map and its derivative. It is easy to derive the Poincaré map and its derivative in the low-dimensional impact oscillators. But, in the high-dimensional impact oscillators [1], derivation and its implementation processes of them will be very complicated. Thus, we believe that the proposed method, which do not require the Poincaré map and its derivative, will be an effective stability analysis method for the impact oscillators.

## 4. Conclusion

In this study, we have proposed a stability analysis method for the two-dimensional linear impact oscillators with periodic border. The proposed method has simple algorithm compared with the previous method [5, 6], and the accuracy of calculation of the proposed method can be said good. So, we conclude that the proposed method, will be an effective stability analysis method for the impact oscillators. The future work is to improve the method for the $n$-dimensional impact oscillators.

## References


[1] C. N. Bapat, "The General Motion of an Inclined Impact Damper with Friction," Journal of Sound and Vibration, vol. 183, no. 3, pp. 417-427, 1995.

[2] T. E. Rook, R. Singh, "Dynamic Analysis of a Reverse-idler Gear Pair with Concurrent Clearances," Journal of Sound and Vibration, vol. 182, no. 2, pp. 303-322, 1995.

[3] E. M. Izhikevich, "Simple Model of Spiking Neurons," IEEE Trans. Neural Netw., vol. 14, no. 6, pp. 1569-1572, 2003.

[4] S. Maggi, S. Rinaldi, "A Second-order Impact Model for Forest Fire Regimes," Theoretical Population Biology, vol. 70, no. 2, pp. 174-182, 2006.

[5] M. Bernardo, C. J. Budd, A. R. Champneys, P. Kowalczyk, "Piecewise-smooth dynamical systems" Springer, 2008.

[6] G. Ikeda, H. Asahara, K. Aihara, T. Kousaka, "Derivation Method of the Bifurcation Point for the Periodic Solution in an Impact Oscillator with Periodic Local Cross-Section," International Symposium on Nonlinear Theory and its Applications, pp. 891-894, 2012.

[7] Y. Yoshitake, A. Sueoka, T. Miyuki, T. Hamano, S. Tamura, "Analysis of Impacting Systems by Shooting Method" Proc. of D & D, p. 66, 2000. (In Japanese)

[8] A. F. Filippov, "Differential Equations With Discontinuous Righthand Sides" Dordrecht, The Netherlands: Kluwer, 1988.

[9] A. B. Nordmark and P. T. Piiroinen, "Simulation and stability analysis of impacting systems with complete chattering," Nonlinear Dynamics, Vol. 58, No.1-2, pp. 85-106, 2009.

[10] S. Kawamura, K. Kitajo, S. Horita, M. Yoshizawa, "Fundamental Study on Impact Oscillations of Rigid Trolley-Pantograph System," The Japan Society of Mechanical Engineers C, vol. 73, no. 728, pp. 974-981, 2007. (In Japanese)

[11] H. Asahara, T. Kousaka, "Stability Analysis for the Interrupted Dynamical Systems," IEEE Workshop on Nonlinear Circuit Networks, pp. 37-38, 2012.

[12] G. Floquet, "Sur les équations differentielles linéaires á coefficients périodiques" Ann. Sci. Ecole Norm. Sup., Ser. 2, 12, p. 47, 1883.