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Application of a Bivariate Fractal Interpolation Surface to an Analysis of Perspective Painting Images

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Abstract—In this paper, a lemma and two theorems for a bivariate fractal interpolation surface (BFIS) generated by an iterated function system (IFS) with an individual vertical scaling factor are introduced. Using an affine transformation and the BFIS, a foundation of analysis of perspective painted images, including images from an arbitrary vantage point together with different sfumato effects, is provided. Finally, this analysis of perspective for painted images is applied to the Virgin section of *Annunciation* by Leonardo da Vinci, and followed by concluding remarks.

1. Introduction

In [1], a lemma and two theorems for a bivariate fractal interpolation surface (BFIS) generated by an iterated function system (IFS) with an individual vertical scaling factor, are provided. For broader application, use of the individual vertical scaling factor for the BFIS is emphasized.

Computer image analysis, such as image warping methods [2], [3] can help art historians to answer much-debated questions; for example, is the geometry of Masaccio's *Trinity* correct, or what is the shape of the dome in Raphael's *School of Athens*? Other interesting questions about the spatial structure of paintings (e.g. linear perspective, anamorphosis, aerial or atmospheric perspective, and sfumato used "to tone down" in a painting or drawing [4]) can also be addressed. Since interpolation methods are indispensable for image warping, the BFIS with an individual vertical scaling factor is expected to be effective for such computer image analysis. In this paper, we consider applying the BFIS to analysis of perspective paintings, such as linear perspective, and sfumato that is often used in connection with the works of Leonardo da Vinci. For purpose of analysis, we focus upon Leonardo's *Annunciation* [7], [8], [9].

First, we introduce a lemma and two theorems for an IFS generating a BFIS with the individual vertical scaling factor shown in [1]. Secondly, we provide a foundation of analysis for perspective painting images, including images from an arbitrary point of view together with different sfumato effects, using an affine transformation and the BFIS. We then apply this analysis of perspective painted images to a key section of *Annunciation*, specifically the area containing the Virgin. Finally, we summarize the concluding remarks.

In the following, let $\boldsymbol{Z}$ be a set of integer numbers, $\boldsymbol{Z}_+$ be a set of non-negative integer numbers, and $\boldsymbol{R}$ be a set of real numbers.

2. Fractal interpolation surface on the rectangular grid points

This section introduces the lemma and two theorems for a bivariate fractal interpolation surface (BFIS) generated by an iterated function system (IFS) with the individual vertical scaling factor shown in [1].

Let $I = [0,1] \subset \boldsymbol{R}$ and $D = [0,1]^2 \subset \boldsymbol{R}^2$. And let $N > 1, M > 1,\ N, M \in \boldsymbol{Z}_+$ be two positive integers. Set $x_i = \frac{i}{N},\ y_j = \frac{j}{M},\ i = 0, \ldots, N,\ j = 0, \ldots, M$. We obtain a set of rectangular grid points $\{(x_i, y_j) \in \boldsymbol{R}^2 | i = 0, \ldots, N,\ j = 0, \ldots, M\}$ of D. Let $\psi(x, y) : D \subset \boldsymbol{R}^2 \to \boldsymbol{R}$ be a continuous function defined on D, and $z_{i,j} = \psi(x_i, y_j)$. Then

$$\{(x_i, y_j, z_{i,j}) \in \boldsymbol{R}^3 | i = 0, \ldots, N,\ j = 0, \ldots, M\} \quad (1)$$

is a set of data points (interpolating points) in $\boldsymbol{R}^3$.

Let $(u_i, v_j)^T : D \to [x_{i-1}, x_i] \times [y_{j-1}, y_j] \subset D, i = 1, \ldots, N,\ j = 1, \ldots, M$, as follows:

$$\begin{pmatrix} u_i(x) \\ v_j(y) \end{pmatrix} = \begin{pmatrix} \frac{(-1)^{\sigma(i+1)}}{N} x + \frac{i-\sigma(i)}{N} \\ \frac{(-1)^{\sigma(j+1)}}{M} y + \frac{j-\sigma(j)}{M} \end{pmatrix} \quad (2)$$

where $\sigma(k) = k \ mod\ 2$.

Let $f_{ij}(x, y) : [x_{i-1}, x_i] \times [y_{j-1}, y_j] \subset D \to \boldsymbol{R},\ i = 1, \ldots, N,\ j = 1, \ldots, M$, be the piecewise polynomial interpolation function, as follows:

$$f_{ij}(x, y) = \sum_{(k,l) \in \{i-1,i\} \times \{j-1,j\}} z_{k,l} \cdot \Phi_{\Delta_{k,l}}(u_i(x)^{-1}, v_j(y)^{-1}) \quad (3)$$

where $\Delta_{k,l} = (N\sigma(k), M\sigma(l))$. The bivariate functions $\Phi_{N,M}, \Phi_{0,0}, \Phi_{0,M}, \Phi_{N,0} : D \to I$ are defined as

$$\Phi_{N,M}(x, y) = xy,\ \Phi_{0,0}(x, y) = (1 - x)(1 - y),$$
$$\Phi_{0,M}(x, y) = (1 - x)y,\ \Phi_{N,0}(x, y) = x(1 - y).$$

Define mappings $w_{ij} : D \times \boldsymbol{R} \to \boldsymbol{R}^3, i = 1, \ldots, N, j = 1, \ldots, M$, as follows:

$$w_{ij}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u_i(x) \\ v_j(y) \\ h_{ij}(x, y, z, d_{ij}) \end{pmatrix},$$
$$= \begin{pmatrix} u_i(x) \\ v_j(y) \\ f_{ij}(u_i(x), v_j(y)) + g_{ij}(u_i(x), v_j(y), z, d_{ij}) \end{pmatrix}, \quad (4)$$

where d_{ij} is a free parameter, which is considered the vertical scaling factor. Here, $g_{ij}(x, y, z, d_{ij}) : [x_{i-1}, x_i] \times [y_{j-1}, y_j] \subset D \times \boldsymbol{R} \times [0, 1] \to \boldsymbol{R}$, $i = 1, \ldots, N$, $j = 1, \ldots, M$, is described, as follows:

$$g_{ij}(x, y, z, d_{ij}) = d_{ij} z - \sum_{(k,l) \in \{i-1,i\} \times \{j-1,j\}} d_{ij} z_{\Delta_{k,l}} \cdot \Phi_{\Delta_{k,l}}(u_i(x)^{-1}, v_j(y)^{-1}), \quad (5)$$

Lemma 1 *$h_{ij}(x, y, z, d_{ij}) : D \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$, $i = 1, \ldots, N$, $j = 1, \ldots, M$, has the following relations such that*

$$|h_{ij}(\xi_1, y, z, d_{ij}) - h_{ij}(\xi_2, y, z, d_{ij})| \leq c \cdot |\xi_1 - \xi_2|, \textit{for } \xi_1, \xi_2 \in I, y \in I, z \in \boldsymbol{R}, \quad (6)$$
$$|h_{ij}(x, \eta_1, z, d_{ij}) - h_{ij}(x, \eta_2, z, d_{ij})| \leq c \cdot |\eta_1 - \eta_2|, \textit{for } \eta_1, \eta_2 \in I, x \in I, z \in \boldsymbol{R}, \quad (7)$$
$$c_{ij} = 4 \cdot \max_{(k,l) \in \{i-1,i\} \times \{j-1,j\}} \{|z_{k,l} - d_{ij} z_{\Delta_{k,l}}|\},$$
$$c = \max_{1 \leq i \leq N, 1 \leq j \leq M} c_{ij},$$
$$|h_{ij}(x, y, z_1, d_{ij}) - h_{ij}(x, y, z_2, d_{ij})| \leq d_{ij} \cdot |z_1 - z_2|, \textit{for } x \in I, y \in I, z_1, z_2 \in \boldsymbol{R}, \quad (8)$$
$$0 \leq d_{ij} < 1.$$

Theorem 1 (IFS Theorem) *Let $N > 1, M > 1$, $N, M \in \boldsymbol{Z}_+$ be two positive integers. Let $\{D \times \boldsymbol{R}; w_{11}, w_{21}, \ldots, w_{NM}\}$ denote the mappings (or IFS) of Eq.(4), associated with the data set of Eq.(1). Let the vertical scaling factor d_{ij} obey $0 \leq |d_{ij}| < 1$ for $i = 1, \ldots, N$, $j = 1, \ldots, M$. Then there is a metric d_m on $\boldsymbol{R}^3$, equivalent to the Euclidean metric, such that IFS of Eq.(4) is hyperbolic with respect to d_m. In particular, there is a unique non-empty compact set, or an attractor: $G \subset \boldsymbol{R}^3$ such that*

$$G = \bigcup_{i,j}^{N,M} w_{ij}(G) \quad (9)$$

.

Theorem 2 (FIS Theorem) *Let $N > 1, M > 1$, $N, M \in \boldsymbol{Z}_+$ be two positive integers. Let $\{D \times \boldsymbol{R}; w_{11}, w_{21}, \ldots, w_{NM}\}$ denote the mappings (or IFS) of Eq.(4), associated with the data set of Eq.(1). Let the vertical scaling factor d_{ij} obey $0 \leq |d_{ij}| < 1$ for $i = 1, \ldots, N$, $j = 1, \ldots, M$, so that the IFS of Eq.(4) is hyperbolic. Let G denote the attractor of the IFS. $\tilde{G}$ is the graph of a continuous function of $\psi(x, y)$: $D \subset \boldsymbol{R}^2 \to \boldsymbol{R}$, which interpolates the data of Eq.(1). That is,*

$$\tilde{G} = ((x, y), \psi(x, y)) : (x, y) \in D, \quad (10)$$
$$\textit{where } \psi(x_i, y_j) = z_{i,j},$$
$$\textit{for } i = 1, \ldots, N, \; j = 1, \ldots, M. \quad (11)$$

1. *If any i, j, d_{ij} are identical to each other; i.e. $d = d_{ij}$, then $G = \tilde{G}$.*
2. *The attractor G of Eq.(4) includes the set of data points of Eq.(1).*
3. *The attractor G of Eq.(4) depends continuously on d_{ij}.*

3. Analysis of a perspective painting image

3.1. Leonardo's *Annunciation* in perspective [8], [9]

Annunciation (in Italian, *Annunciazione*) by Leonardo da Vinci at the Uffizi Gallery is well known to be rigorously constructed according to linear perspective rules. If we look at the painting from a central vantage point, however, the Virgin Mary's arm seems to be too long and dislocated at certain points. Some academics tried to attribute this evident imperfection to the inexperience of the young artist. Based on an idea originating with Carlo Pedretti Antonio Natali (Director of the Uffizi) some academics supposed that the painting was originally conceived to be observed from a different vantage point. If we look at it from another point, on the right side and a little from below (i. e. from a forty-five degree angle to the right), we can see a normal arm perfectly placed. It is pointed out that the Virgin anomaly is an example of anamorphosis by Leonardo. The garden in which the annunciation scene takes place displays several botanical species that are represented according to an aerial perspective reproducing human sight then gradually obstructed by atmospheric dust. This forewarns of the sfumato, Leonardo's famous technique.

3.2. Affine transformation [6], pp. 49 - 53

To obtain an image of the painting from an arbitrary vantage point, we define an affine map $\boldsymbol{A}$ as follows:

$$\boldsymbol{A}\begin{bmatrix} x^0 \\ y^0 \end{bmatrix} = \begin{bmatrix} r_1 \cos\theta_1 & r_1 \sin\theta_1 \\ -r_2 \sin\theta_2 & r_2 \cos\theta_2 \end{bmatrix}\begin{bmatrix} x^0 \\ y^0 \end{bmatrix} \quad (12)$$

Here let $N_0 > 1, M_0 > 1$, $N_0, M_0 \in \boldsymbol{Z}_+$ be two positive integers. We obtain another set of rectangular grid points $\{(x_i^0, y_j^0) \in \boldsymbol{R}^2 | i = 0, \ldots, N_0, j = 0, \ldots, M_0\}$ of D. Let $\phi(x^0, y^0) : D \subset \boldsymbol{R}^2 \to \boldsymbol{R}$ be a continuous function defined on D, and $z_{i,j}^0 = \phi(x_i^0, y_j^0)$. Then

$$\{(x_i^0, y_j^0, z_{i,j}^0) \in \boldsymbol{R}^3 | i = 0, \ldots, N_0, j = 0, \ldots, M_0\} \quad (13)$$

is a set of data points (interpolating points) in $\boldsymbol{R}^3$.

If the source image ϕ of Eq.(13) defined over a (x^0, y^0) coordinate system undergoes an affine transformation (i. e. linear perspective) from an arbitrary vantage point to produce a destination image ψ of Eq.(1) defined over an (x, y) coordinate system, this affine transformation (of the coordinates) is expressed as Eq.(12):

$$[x, y]^T = \boldsymbol{A} \left[x^0, y^0\right]^T . \tag{14}$$

3.3. The images from another vantage point together with different sfumato effects

Subsequently, we focused on the right area of *Annunciation*, specifically the Virgin section of the *Annunciation* image. The source image ϕ : the Virgin section of the *Annunciation* image, and ϕ' : the edge image of ϕ are illustrated in Fig. 1, and Fig. 3, respectively. Using the affine transformation of Eq.(14) with the parameter : $r_1 = 1, r_2 = \frac{47}{50}$, $\theta_1 = \frac{\pi}{180}$, and $\theta_2 = 0$, we identify the destination images from about a forty-five degree angle to the right, ψ, 256^3 RGB levels of Fig. 2, and ψ' from the source images ϕ, 256^3 RGB levels of Fig. 1 and ϕ', 256 gray levels of Fig. 3. Further using the mappings of Eq.(4), and an algorithm similar to Algorithm 8.2 [6], pp.84 - 91, on MATLAB, we illustrate the examples of sfumato effects for the painted images, 256^3 RGB levels, size (N', M'), from the modified image ψ, size (N, M): $(N, M) \rightarrow (N', M')$. From the FIS generated by Eq.(4), the number of iterations: 3×10^5, and the composite images of the Virgin consisting of Red, Green, and Blue color reconstructed images, are illustrated in Fig 4 : ($d_{ij} = 0.3$, for the edge of ψ'), and in Fig. 5 : ($d_{ij} = 0.09$, for the edge of ψ'). In each color image, $d_{ij} = 0.03$ for the other region except for the edge of ψ', respectively. Comparing the viewpoints and the sfumato effects between Fig. 1, Fig. 4, and Fig. 5, we can see the difference in the Virgin's facial expressions. For purposes of comparison, an existing technology with similar properties to those of this study include "atmospheric perspective effect enhancement" [5]. A study of the same painting would be the ideal; however, the authors were unable to discover an analysis of *Annunciation* using this technology. On the other hand, a research study, *Atmospheric Perspective Effect Enhancement of Landscape Photographs Through Depth-Aware Contrast Manipulation*[5], was accessed and provided data which could be used for a comparison (see Table 1).

4. Concluding Remarks

1. We have introduced the lemma and theorems to establish the BFIS with an individual vertical scaling factor for every set of rectangular grid data, and have provided the foundation for analysis of perspective painted images, including images from arbitrary vantage points together with different sfumato effects, using an affine transformation and the BFIS.
2. We have shown examples of the sfumato effects, in particular, in Fig. 4 and in Fig. 5. The effect of the individual vertical scaling factor (i.e. the strength of the sfumato) was demonstrated such that the eyes and the mouth in Fig. 5 are more chiseled than those in Fig. 4, when magnified.

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Table 1: Comparison of our work and an existing technology [5]

	this study	depth-aware contrast manipulation
contrast	individual vertical scaling factor control	inter-contrast and intra-contrast control

Figure 1: The Virgin section of *Annunciation* image ϕ : size $(N_0, M_0) = (2031, 1856)$

Figure 2: The Virgin section of *Annunciation* image ψ : size $(N_0, M_0) = (2031, 1892)$

Figure 3: Edge image ϕ' of Fig. 1 : size $(N_0, M_0) = (2031, 1856)$

Figure 4: A reconstruction of the Virgin image from the FIS (1) : size $(N', M') = (2031, 1892)$. The lightest sfumato is used to effect the graininess of the figure (NB apart from the face).

Figure 5: A reconstruction of the Virgin image from the FIS (2) : size $(N', M') = (2031, 1892)$. The second lightest sfumato is used to effect the graininess of the figure (NB apart from the face).