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Synchronized bifurcation in a two-coupled Izhikevich neuron model

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Abstract—In a coupled Izhikevich neuron model, there are some parameter regions where each neuron arises firing at the same time. We observed discontinuous changes of stability of the solutions in these regions. In this study, we regard these changes as a bifurcation phenomena. We present their examples and condition of the bifurcation. Finally we calculate sets of the bifurcation.

1. Introduction

Izhikevich neuron model[1] can present a lot of general firing patterns, which is observed in real neurons. In addition, the model can keep the costs of calculation lower than the other models. From these reasons, this model is well used in many researches. For example, Tamura[2] proposed the first result of bifurcation analysis for the model. On the other hands, Ito[3] suggested a method of bifurcation analysis for two-coupled Izhikevich neuron model. The study[3] is important for the application such that the neural network. They tried to avoid matching of the two timing. One is the time at firing and the other is the time at the Poincaré map. In this study, we focus on this matching. In coupled Izhikevich neuron model, firings of each neuron synchronize. The phenomenon has not been observed in [3] and any other researches. We found that the phenomenon makes quantitative changes of ω-limit set of the system. That is, the phenomenon can be regarded as a global bifurcation phenomenon. We call this phenomenon as "synchronized bifurcation" in this study. This study shows the characteristics of this study, the method how we calculate the bifurcation sets and the changes how it makes to the ω-limit set of this system.

2. Izhikevich neuron model and its coupled system

The neuron model proposed by Izhikevich[1] is given by

$$\begin{cases} \dfrac{dv}{dt} &= 0.04v^2 + 5v + 140 - u + I \\ \dfrac{du}{dt} &= a(bv - u) \end{cases} \tag{1}$$

where, v and u are state variables and a, b, I and δ are parameters. Firing phenomena are realized by following maps:

$$\text{if } v \leq 30, \text{ then } \begin{cases} v & \mapsto & c \\ u & \mapsto & u + d \end{cases}, \tag{2}$$

where, c measures a voltage after firing and d measures strength of the restoration.

Gap-junction two-coupled Izhikevich neuron model is given by

$$\begin{aligned} \frac{d\boldsymbol{v}}{dt} &= \boldsymbol{f}(\boldsymbol{v}, a_1, \delta) \\ &= \begin{pmatrix} 0.04v_1^2 + 5v_1 + 140 - u_1 + I - \delta(v_2 - v_1) \\ a_1(b_1 v_1 - u_1) \\ 0.04v_2^2 + 5v_2 + 140 - u_2 + I - \delta(v_1 - v_2) \\ a_2(b_2 v_2 - u_2) \end{pmatrix} \end{aligned} \tag{3}$$

where, $\boldsymbol{v} = (v_1, u_1, v_2, u_2)$ is a vector for state variable and δ is strength of the junction. For each neuron, firings arise with the following condition:

$$\text{if } v_i \leq 30, \text{ then } \begin{cases} v_i & \mapsto & c \\ u_i & \mapsto & u_i + d \end{cases}, \quad i = 1, 2 \ . \tag{4}$$

On this study a_1 and δ are variable and the other are static.

$$a_2 = 0.2, b_1 = 0.2, b_2 = 0.2, c = -50, d = 2, I = 10. \tag{5}$$

Figure 1 shows some time waves of system (3), where, t_i is the time when v_i fires. Figure 1(a) shows a stable periodic solution whose $t_1 > t_2$. Through undergoing the situation $t_1 = t_2$: synchronized as Fig.1(b), firing order changes as $t_1 < t_2$. Then stability of the solution becomes unstable. That is, stability of the solution immediately changes since undergoing the synchronized firing.

3. Changing of stability of periodic solution

To evaluate the stability of periodic solutions in this system, the method proposed by Kousaka[4] is strongly effective. The method[4] can solve the bifurcation problem of hybrid system. Hybrid system has digital states (*modes*) and analog states (*states*) in its structure. Each mode transits immediately and discontinuously from one to the other(s). These changes called mode transition. Each state evolves by time-continuous or time discrete dynamical system(s).

From the result of previous study[3], let us define the Poincaré section as follows:

$$\Pi_0 = \left\{ \boldsymbol{v} = (v_1, u_1, v_2, u_2) \in R^4 \mid q(\boldsymbol{v}) = v_1 = 0 \right\}. \tag{6}$$

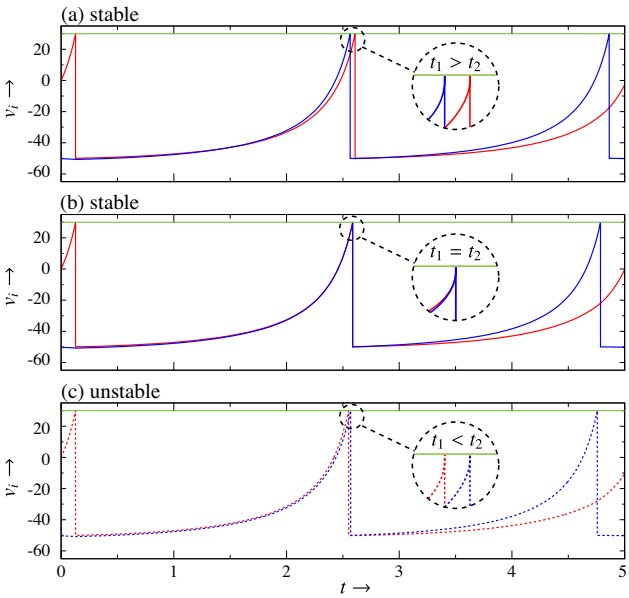

Figure 1: Time waves of v_1(red) and v_2(blue) with (a) $a_1 = 0.168, \delta = 0.09$: stable 2-periodic solution, (b) $a_1 = 0.17, \delta = 0.09$: synchronized 2-periodic solution and (c) $a_1 = 0.171, \delta = 0.09$: unstable 2-periodic solution.

Let us consider the solutions start from $\boldsymbol{v}_0$ on Π_0 and return to a state on Π_0 via m-times mode transition. $q_k(\boldsymbol{v}) = 0$ is a condition equation for k-times mode transition and $\Pi_k = \{\boldsymbol{v} \mid q_k(\boldsymbol{v}) = 0\}$ is a manifold expanded by $q_k(\boldsymbol{v}) = 0$. The solution starting from $\boldsymbol{v}_k \in \Pi_k$ at the time t_k is

$$\boldsymbol{v}_k(t) = \boldsymbol{\varphi}_k(t, \boldsymbol{v}_k, t_k, \lambda), \tag{7}$$

where, λ is a certain parameter and

$$\boldsymbol{v}_k(t_k) = \boldsymbol{v}_k = \boldsymbol{\varphi}_k(t_k, \boldsymbol{v}_k, t_k, \lambda). \tag{8}$$

Local map from Π_k to Π_{k+1} is

$$\begin{aligned} T_k \quad : \quad & \Pi_k \rightarrow \Pi_{k+1}, \\ & \boldsymbol{v}_k \mapsto \boldsymbol{v}_{k+1} = T_k(\boldsymbol{v}_k) = \boldsymbol{\varphi}_k(t_{k+1}, \boldsymbol{v}_k, t_k, \lambda). \end{aligned} \tag{9}$$

From Eq.(9), the Poincaré map T is expanded as

$$T = T_{m-1} \circ \ldots \circ T_0(\boldsymbol{v}_0). \tag{10}$$

Especially when

$$T^l(\boldsymbol{v}_0) - \boldsymbol{v}_0 = 0, \tag{11}$$

the solution v_0 is called as l-periodic solution. The Poincaré map T has some trivial factors related to a normal vector of Π_0. To exclude the factors, let us define local coordinate system $\Sigma \subset R^{n-1}$ and local coordinate $\boldsymbol{u} = (u_1, v_2, u_2) \in \Sigma$.

$$p^{-1} : \begin{array}{ccc} \Sigma & \rightarrow & \Pi_0 \\ \boldsymbol{u} & \mapsto & \boldsymbol{v} \end{array}, \quad p : \begin{array}{ccc} \Pi_0 & \rightarrow & \Sigma \\ \boldsymbol{v} & \mapsto & \boldsymbol{u} \end{array}. \tag{12}$$

The Poincaré map on Σ is give by

$$\begin{aligned} T_\ell \quad : \quad & \Sigma \rightarrow \Sigma \\ & \boldsymbol{u} \mapsto p \circ T \circ p^{-1} \end{aligned}. \tag{13}$$

On local coordinate system, Eq.(11) is

$$\boldsymbol{u}_0 = T_\ell(\boldsymbol{u}_0) = p \circ T \circ p^{-1}(\boldsymbol{u}_0) \tag{14}$$

The derivative of the Poincaré map T with respect to the initial value $\boldsymbol{v}_0$ is

$$DT = \frac{\partial T_\ell}{\partial \boldsymbol{u}_0} = \frac{\partial p}{\partial \boldsymbol{v}} \frac{\partial T}{\partial \boldsymbol{v}_0} \frac{\partial p^{-1}}{\partial \boldsymbol{u}_0} = \frac{\partial p}{\partial \boldsymbol{v}} \prod_{k=0}^{m-1} \frac{\partial T_k}{\partial \boldsymbol{v}_k} \frac{\partial p^{-1}}{\partial \boldsymbol{u}_0}. \tag{15}$$

Each derivative is given by

$$\frac{\partial T_k}{\partial \boldsymbol{v}_k} = \left[I - \frac{1}{\frac{\partial q_{k+1}}{\partial \boldsymbol{v}} \boldsymbol{f}} \boldsymbol{f} \frac{\partial q_{k+1}}{\partial \boldsymbol{v}} \right] \frac{\partial \boldsymbol{\varphi}_k}{\partial \boldsymbol{v}_k} \tag{16}$$

$$\frac{\partial p}{\partial \boldsymbol{v}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{17}$$

$$\frac{\partial p^{-1}}{\partial \boldsymbol{u}_0} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{18}$$

where, I is a 4×4 identity matrix. $\partial \boldsymbol{\varphi}_k / \partial \boldsymbol{v}_k$ is derived by following ordinary differential equation:

$$\frac{d}{dt} \frac{\partial \boldsymbol{\varphi}_k}{\partial \boldsymbol{v}_k} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{v}} \frac{\partial \boldsymbol{\varphi}_k}{\partial \boldsymbol{v}_k} \text{ with } \left. \frac{\partial \boldsymbol{\varphi}_k}{\partial \boldsymbol{v}_k} \right|_{t=t_k} = I. \tag{19}$$

The characteristic equation is given by

$$\chi(\mu_j) = \det\left(DT - \mu_j I\right) = 0, \quad j = 1, 2, 3, \tag{20}$$

where, μ_j is characteristic multiplier, which measures the stability of the Poincaré map T. That is, μ_j can be an index of stability of a periodic solution. When $^\forall j$, $|\mu_j| < 1$, the solution is stable. When $^\exists j$, $|\mu_j| > 1$, the solution is unstable.

Figure 2 shows a root locus that presents how characteristic multiplier changes by undergoing the synchronized firing. Each of Fig.2(a)-(b) denotes that the synchronized firing changes values of all μ_j immediately and discontinuously. Especially for (a), changes of μ_j affect the stability of solution since μ_j goes between the regions where $|\mu_j| > 1$ and $|\mu_j| < 1$ via the synchronized firing. We call this case as *synchronized bifurcation*.

When we focus on the firing order and the product operation of matrices, the cause of these phenomena is unveiled. DT of the solutions shown on Fig. 1 is expanded as

$$\frac{\partial T}{\partial \boldsymbol{v}_0} = \frac{\partial T_5}{\partial \boldsymbol{v}_5} \frac{\partial T_4}{\partial \boldsymbol{v}_4} \frac{\partial T_3}{\partial \boldsymbol{v}_3} \frac{\partial T_2}{\partial \boldsymbol{v}_2} \frac{\partial T_1}{\partial \boldsymbol{v}_1} \frac{\partial T_0}{\partial \boldsymbol{v}_0}. \tag{21}$$

For the case Fig.1(a),

$$\frac{\partial T_3}{\partial \boldsymbol{v}_3} \frac{\partial T_2}{\partial \boldsymbol{v}_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.004 & 0.997 & 0 & 0 \\ 0.005 & 0 & 1.021 & -0.019 \\ 0.005 & 0 & 0.001 & 0.9962 \end{pmatrix}$$

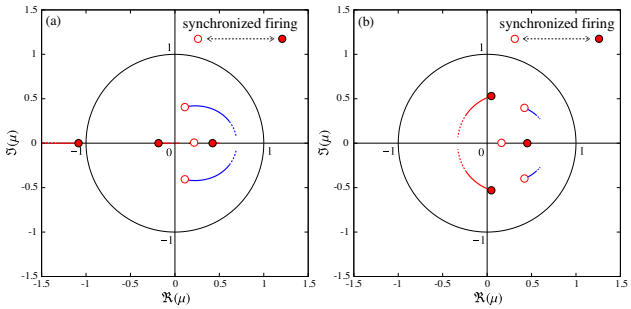

Figure 2: Root locus via the synchronized firing on complex plane. Red curves: firing order of the solution is $v_1 \to v_1 \to v_2 \to v_2$. Blue curves: firing order of the solution is $v_1 \to v_2 \to v_1 \to v_2$. (a) $a_1 \in (0.159, 0.187), \delta = 0.1$, (b) $a_1 \in (0.168, 0.174), \delta = 0.09$

$$\begin{pmatrix} 1.972 & -0.163 & -1.778 & 0.144 \\ 0.005 & 0.982 & -0.008 & 0.001 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.006 & 0.9785 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.003 & 0.979 & 0 & 0 \\ 0.033 & -0.003 & -0.029 & -0.016 \\ 0.01 & -0.001 & -0.015 & 0.976 \end{pmatrix},$$

on the other hand for Fig.1(c),

$$\frac{\partial T_3}{\partial \boldsymbol{v}_3}\frac{\partial T_2}{\partial \boldsymbol{v}_2} = \begin{pmatrix} 1.253 & -0.232 & 0.013 & -0.003 \\ 0.007 & 0.966 & 0.011 & -0.001 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.011 & 0.9709 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.005 & 0.980 & 0 & 0 \\ -0.681 & 0.061 & 1.625 & -0.166 \\ -0.008 & 0.001 & 0.007 & 0.974 \end{pmatrix}$$
$$= \begin{pmatrix} -0.008 & -0.226 & 0.021 & -0.005 \\ -0.012 & 0.947 & 0.018 & -0.003 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.011 & 0.938 \end{pmatrix}.$$

A change of firing order affects the order of line of matrix, that is, the line constructed only by "0" changes its order. Since matrix operation does not have commutative, the change causes not negligible changes on the calculation results.

4. Bifurcation analysis

Condition equation of the synchronized bifurcation is given by

$$t_1 - t_2 = 0. \tag{22}$$

Thus, by solving Eq. (14) and Eq. (22) at the same time, synchronized bifurcation sets of l-periodic solution is obtained. For obtaining the bifurcation set, the Newton's method is effective. The method needs DT and $\partial t_i/\partial \boldsymbol{u}_0$ for calculation. This factor is derived by previous method[5].

The local bifurcation sets are obtained by solving Eq. (20) with a condition $|\mu_j| = 1$.

Figure 3 shows the result of bifurcation analysis and Fig.4 presents phase portraits on each points of Fig.3. At point (a) on Fig.3, there is a stable 1-periodic solution shown on Fig.4. By undergoing period-doubling bifurcation I^1, the solution becomes 2-periodic solution shown on Fig.4(b) at point (b). When parameters are set at point (c), there are no stable 2-periodic solutions and we can observe chaos shown on Fig.4(c). On the region including the point (d), we can observe a 2-periodic solution shown as Fig. 4(d). When seeing v_1-u_1 plane, the solution has similar structure to the solution on Fig. 4(b). When seeing v_1-v_2 plane, the solutions are exactly different. This is because the order of firing has changed between these two solutions. The changing is caused by SF^2 in Fig. 3.

5. Conclusion

This study investigated the synchronized firing phenomena(SF) observed in 2-coupled Izhikevich neuron model. In our research,

- a drastic changing of index for stability of a periodic solution has arose via SF,
- SF has sometimes made change the stability of a periodic solution: stable to unstable(and vice versa),
- condition of arising SF has been derived,
- a set of parameters where SF arises has been obtained.

For future work, we should try to confirm SF in 3 or more coupled neuron model.

References

[1] E. M. Izhikevich et al., "Simple model of spiking neurons," IEEE Trans. neural networks, vol. 14, no. 6, pp. 15691572, 2003.

[2] A. Tamura, T. Ueta and S. Tsuji, "Bifurcation Analysis of Izhikevich Neuron Model," Dynamics of Continuous, Discrete and Impulsive Systems, 2009.

[3] D. Ito, T. Ueta, and K. Aihara, "Bifurcation analysis of two coupled Izhikevich oscillators," in Proc. NOLTA2010, (Krakow), pp. 627630, Sep. 2010.

[4] T. Kousaka, T. Ueta, and H. Kawakami, "Bifurcation of switched nonlinear dynamical systems," IEEE Trans. Circuits Systs., vol. 46, no. 7, pp. 878885, 1999.

[5] Y. Miino, D. Ito, and T. Ueta, "A computation method for non-autonomous systems with discontinuous characteristics," Chaos, Solitons & Fractals, vol. 77, pp. 277285, 2015.

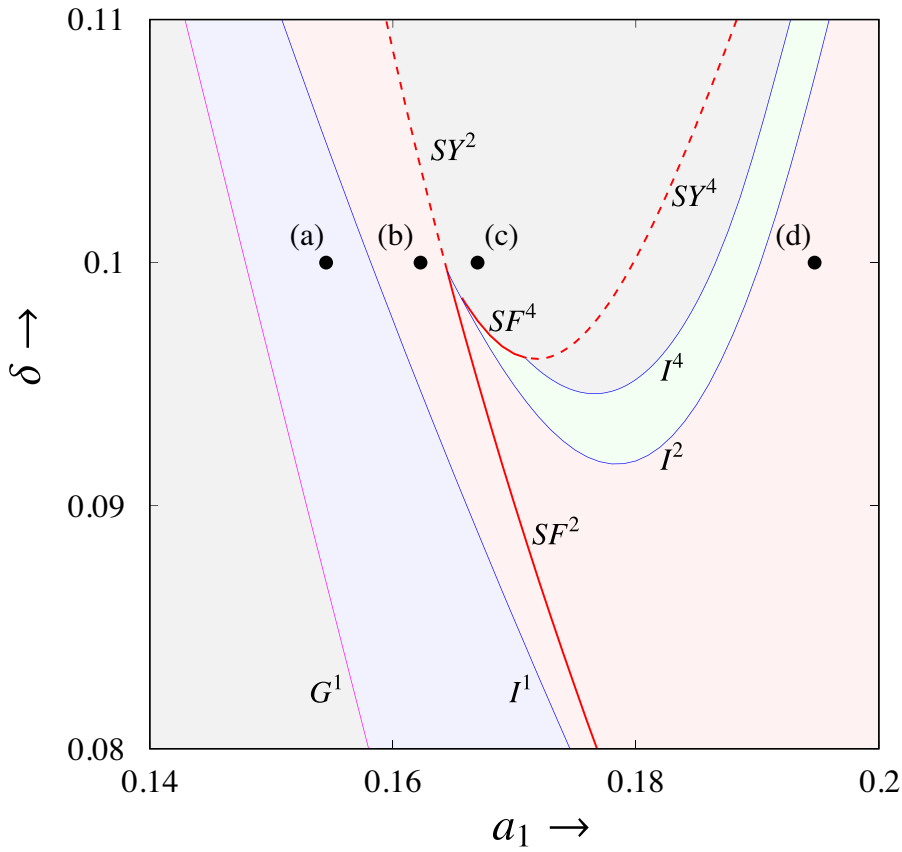

Figure 3: Bifurcation diagram with $a_1 \in (0.14, 0.2), \delta \in (0.08, 0.11)$. Red solid curves are set of synchronized firing. Red broken curves are set of synchronized bifurcation. Grey region presents unstable solutions. Blue region presents 1-periodic solutions. Red region presents 2-periodic solutions. Green region presents 4-periodic solution. G^i means tangent bifurcation, I^i means period-doubling bifurcation and SY^i means synchronized bifurcation, respectively from i-periodic solutions.

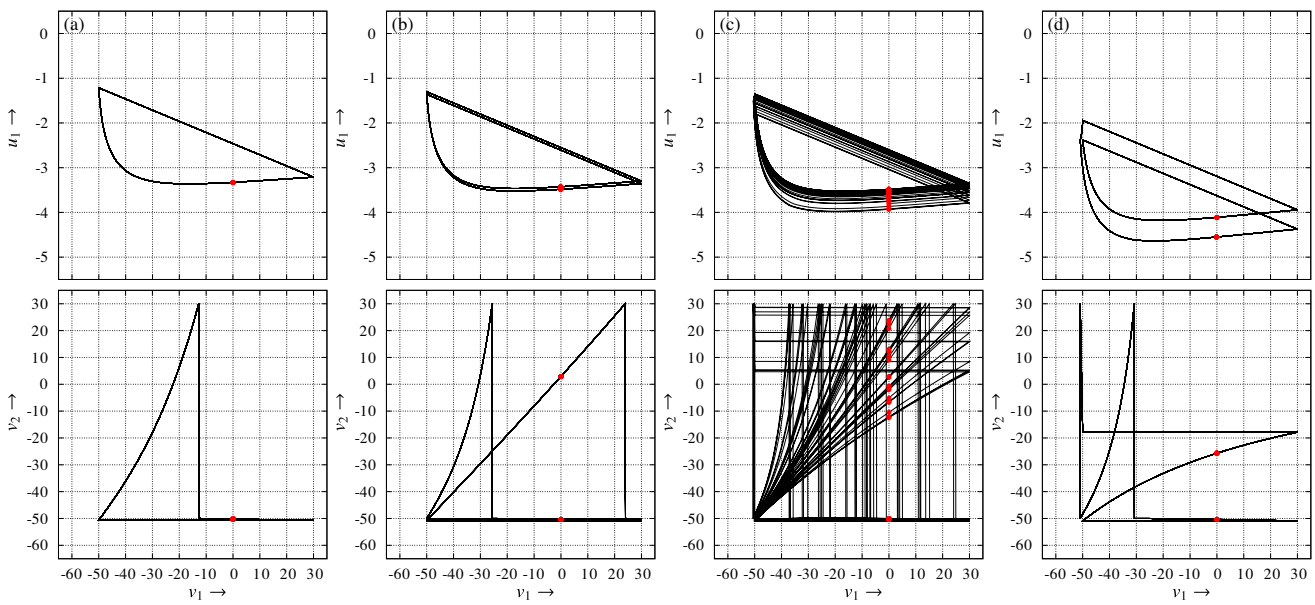

Figure 4: Phase portrait of system (3)–(4) on (top) $v_1 - u_1$ plane and (bottom) $v_1 - v_2$plane. (red points: Poincaré map, $\delta = 0.1$, (a) 1-periodic solution: $a_1 = 0.155$, (b) 2-periodic solution: $a_1 = 0.163$, (c) chaos: $a_1 = 0.168$, (d) 2-periodic solution: $a_1 = 0.195$)