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Development of Discrete Mechanics for 2-dimensional Distributed Parameter Mechanical Systems and Its Application to Vibration Suppression Control of a Film

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Abstract—This paper develops a new discretizing method for 2-dimensional distributed parameter mechanical systems, called "discrete mechanics", and considers its applications to control theory. Especially, a new control method based on discrete mechanics and nonlinear optimization is proposed. The new method is applied to the vibration suppression problem of a film as a physical example. From numerical simulation results, it turns out that vibration of the film is suppressed by control inputs and the whole of the film is stabilized.

1. Introduction

Control of distributed parameter systems is well-known to be one of the most challenging problems in control theory. In general, control of distributed parameter systems is more difficult than that of concentrated constant systems, because distributed parameter systems are represented by infinite-dimensional equations and a lot of actuators and sensors are needed for control. There are generally two kinds of ways to control distributed parameter systems; analytic methods and numerical methods. Especially, numerical methods are powerful tools and a lot of work have been done so far. The authors have been developed a new discretizing method of distributed parameter mechanical systems called "discrete mechanics," which is an extension of the case for concentrated constant systems [1, 2, 3, 4, 5]. The main concept of discrete mechanics is that we first discretize some fundamental concepts of classical physics such as Lagrangian and Hamilton's principal, and then derive discrete euler-Lagrange equations by using discrete Hamilton's principle. In [6, 7], for the 1-dimensional case, a new control method based on discrete mechanics and nonlinear optimization has been proposed and its application potentiality has been confirmed by numerical simulations.

This study aims at development of discrete mechanics for 2-dimensional distributed parameter mechanical systems as an extension of the previous work for the 1-dimensional case [6, 7]. and an application to vibration suppression control of a film. First, Section 2 describes details on discrete mechanics for 2-dimensional distributed parameter mechanical systems. Next, Section 3 presents a new control method via a blending method of discrete mechanics and nonlinear optimization. Then, in Section 4, the vibration suppression control problem of a film is considered as a physical example, and some numerical simulations are shown in order to check the effectiveness of the new method.

2. Discrete Mechanics for 2-dimensional Distributed Parameter Mechanical Systems

This section derives some important concepts on discrete mechanics for 2-dimensional distributed parameter mechanical systems. Let us denote the time variable as $t \in \mathbf{R}$ and the position of the 2-dimensional space as $(x, y) \in \mathbf{R}^2$. We also refer a displacement of the system at the time t and the position (x, y) as $u(t, x, y) \in \mathbf{R}$, and $u(t, x, y)$ with a subscript indicates partial derivative of $u(t, x, y)$ with respect to the subscript, e.g. u_t, u_x, u_y. In this paper, we deal with a continuous Lagrangian density which includes through first-order partial derivative of $u(t, x, y)$ as

$$L^c(t, x, u, u_t, u_x, u_y). \tag{1}$$

Next, we consider discretization of variables. As shown in Fig. 1, the time variable t and the position (x, y) are discretized with sampling intervals h, d_x, and d_y as

$$\begin{aligned} t &\approx h\,k \quad (k = 1, 2, \cdots, K-1, K), \\ x &\approx d_x\,l \quad (l = 1, 2, \cdots, L-1, L), \\ y &\approx d_y\,m \quad (l = 1, 2, \cdots, M-1, M), \end{aligned} \tag{2}$$

where k, l, and m are indices of t, x, and y, respectively.

Now, we use a new notation $U_{k,l,m} \in \mathbf{R}$ as a discrete version of the displacement of the system at the time step k and the position (l, m). Then, we assume that the continuous displacement of the system at the time t and the position (x, y): $u(t, x, y)$ is represented as

$$\begin{aligned} &u(t, x, y) \approx \\ &(1-\alpha)(1-\beta_1)(1-\beta_2)U_{k,l,m} + (1-\alpha)(1-\beta_1)\beta_2 U_{k,l,m+1} \\ &\ + (1-\alpha)\beta_1(1-\beta_2)U_{k,l+1,m} + (1-\alpha)\beta_1\beta_1 U_{k,l+1,m+1} \\ &(\alpha(1-\beta_1)(1-\beta_2)U_{k+1,l} + \alpha(1-\beta_1)\beta_2 U_{k+1,l,m+1} \\ &\ + \alpha\beta_1(1-\beta_2)U_{k+1,l+1,m} + \alpha\beta_1\beta_1 U_{k+1,l+1,m+1} \end{aligned} \tag{3}$$

with 8 displacement variables: $U_{k,l,m}$, $U_{k,l,m+1}$, $U_{k,l+1,m}$, $U_{k,l+1,m+1}$, $U_{k+1,l,m}$, $U_{k+1,l,m+1}$, $U_{k+1,l+1,m}$, $U_{k+1,l+1,m+1}$,

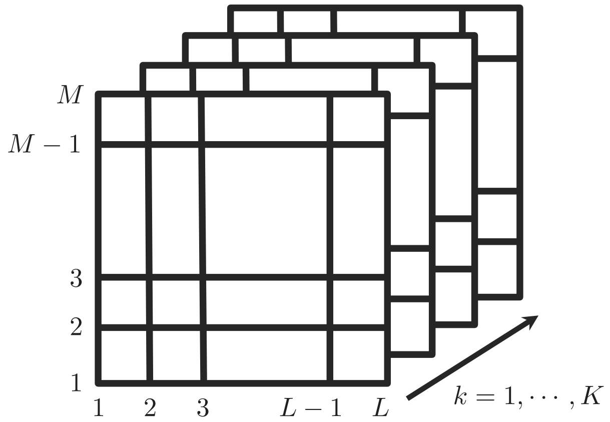

Figure 1: Discretization Setting

where $\alpha,\ \beta_1,\ \beta_2 \in \mathbf{R}$ are dividing parameters ($0 < \alpha,\ \beta_1,\ \beta_2 < 1$). Partial derivatives of $u(t,x,y)$ are also represented by

$$u_t(t,x,y) \approx \frac{U_{k+1,l,m} - U_{k,l,m}}{h},\quad u_x(t,x,y) \approx \frac{U_{k,l+1,m} - U_{k,l,m}}{d_x},\quad u_y(t,x,y) \approx \frac{U_{k,l,m+1} - U_{k,l,m}}{d_y}. \tag{4}$$

By substituting (2)–(4) into (1) and multiplying it by hd_xd_y, we define "a discrete Lagrangian density" as

$$L^d_{k,l,m} \approx hd_xd_yL^c. \tag{5}$$

We also define "a discrete action sum" as

$$S^d(U) := \sum_{k=2}^{K-1}\sum_{l=2}^{L-1}\sum_{m=2}^{M-1} L^d_{k,l,m}, \tag{6}$$

and consider "a discrete variation" as

$$\delta S^d(U) := S^d(U+\delta U) - S^d(U), \tag{7}$$

where δU is a variation of U and satisfies the boundary conditions:

$$\delta U_{1,l,m} = \delta U_{K,l,m} = \delta U_{k,1,m} = \delta U_{k,L,m} = \delta U_{k,l,1} = \delta U_{k,l,M} = 0. \quad (k = 1,\cdots,K;\ l = 1\cdots,L;\ m = 1\cdots,M) \tag{8}$$

As an analogy of Hamilton's principle in the continuous version, we consider "discrete Hamilton's principle" and it states that "only a motion such that the discrete action sum (6) is stationary, that is, $S^d(U) = 0$, can be realized." By applying discrete Hamilton's principle to the discrete action sum (6) and calculating in details, we can derive "discrete Euler-Lagrange equations" as the following (due to limitations of space, the proof is omitted).

Theorem 1 : For the discrete Lagrangian density $L^d_{k,l,m}$ (5) for 2-dimensional distributed parameter mechanical systems, the discrete Euler-Lagrange equation that satisfies discrete Hamilton's principle is given by

$$\frac{\partial L^d_{k-1,l-1,m-1}}{\partial U_{k,l,m}} + \frac{\partial L^d_{k-1,l-1,m}}{\partial U_{k,l,m}} + \frac{\partial L^d_{k-1,l,m-1}}{\partial U_{k,l,m}} + \frac{\partial L^d_{k-1,l,m}}{\partial U_{k,l,m}} + \frac{\partial L^d_{k,l-1,m-1}}{\partial U_{k,l,m}} + \frac{\partial L^d_{k,l-1,m}}{\partial U_{k,l,m}} + \frac{\partial L^d_{k,l,m-1}}{\partial U_{k,l,m}} + \frac{\partial L^d_{k,l,m}}{\partial U_{k,l,m}} = 0 \tag{9}$$

$(k = 2,\cdots,K-1; l = 2\cdots,L-1; m = 2\cdots,M-1)$

We can calculate all the KLM displacements $U_{k,l,m}$ ($1 \le k \le K;\ 1 \le l \le L;\ 1 \le m \le M$) by using the discrete Euler-Lagrange equations (9) under suitable initial and boundary conditions. In addition, the discrete Euler-Lagrange equations (9) are generally nonlinear and implicit, and hence we need some numerical solutions for nonlinear equations such as Newton's method in order to calculate all the displacements of the system.

3. Optimal Control Method via Discrete Mechanics

In this section, a nonlinear control problem for a mathematical model derived by discrete mechanics is formulated, and a solution method of the problem is considered. First, the setting on control inputs is shown. Denote a control input at the time step k and the position (l,m) as $F_{k,l,m} \in \mathbf{R}$. If an actuator is not installed at the position (l,m), we set $F_{k,l,m} \equiv 0$ ($k = 1,\cdots,K$). We also denote a set of indices (l,m) such that actuators are installed as Δ. Thus, the discrete Euler-Lagrange equations with control inputs are given by

$$\frac{\partial L^d_{k-1,l-1,m-1}}{\partial U_{k,l,m}} + \frac{\partial L^d_{k-1,l-1,m}}{\partial U_{k,l,m}} + \frac{\partial L^d_{k-1,l,m-1}}{\partial U_{k,l,m}} + \frac{\partial L^d_{k-1,l,m}}{\partial U_{k,l,m}} + \frac{\partial L^d_{k,l-1,m-1}}{\partial U_{k,l,m}} + \frac{\partial L^d_{k,l-1,m}}{\partial U_{k,l,m}} + \frac{\partial L^d_{k,l,m-1}}{\partial U_{k,l,m}} + \frac{\partial L^d_{k,l,m}}{\partial U_{k,l,m}} = F_{k,l,m} \tag{10}$$

$(k = 2,\cdots,K-1; l = 2\cdots,L-1; m = 2\cdots,M-1).$

In this study, the next control problem is dealt with for the discrete Euler-Lagrange equations with control inputs (10).

Problem 1 : For the discrete Lagrangian density (5) and the discrete Euler-Lagrange equation with control inputs (10), find control inputs $F_{k,l,m}$ ($k = 2,\cdots,K-1$; $(l,m) \in \Delta$) that make all the specified displacements $U_{k,l,m}$ ($k = \kappa,\cdots,K;\ l = 1,\cdots,L,\ m = 1,\cdots,M$) converge to 0. □

In order to solve Problem 1, we consider an optimal control approach. Using weight parameters $a,\ b,\ c$, we set an evaluation function as

$$J(U,F) = a\sum_{k=1}^{\kappa-1}\sum_{l=1}^{L}\sum_{m=1}^{M} U_{k,l,m}^2 + b\sum_{k=\kappa}^{K}\sum_{l=1}^{L}\sum_{m=1}^{M} U_{k,l,m}^2 + c\sum_{k=3}^{K-2}\sum_{(l,m)\in\Delta} F_{k,l,m}^2 \tag{11}$$

where the first and second terms evaluate the displacements from $k = 1$ to $k = \kappa - 1$ and ones from $k = \kappa$ to $k = K$, respectively, and the third term evaluates the values of control inputs. It can be expect that we can make all the specified displacements converge to 0. by minimizing the evaluation function (11). The optimal control problem for the discrete Euler-Lagrange equation with control inputs (10) can be formulated as

$$\begin{aligned} &\min_{U,F}\ (11), \\ &\text{subject to}\ (10), \\ &\text{given}\ \text{initial conditions},\ \text{boundary conditions}. \end{aligned} \tag{12}$$

The optimal control problem (12) can be referred as a finite-dimensional nonlinear optimization problem with constraints, and hence we can solve it by numerical solutions such as “the sequential quadratic programming method” [8]. It is known that the sequential quadratic programming method can be applied to a relatively large-scale problems and effectively obtain an optimal or near-optimal solution.

4. Vibration Suppression Control of Film

This section deals with an application to a physical system “a film,” and confirms the effectiveness of the proposed control method via numerical simulations. It is assumed that the shape of the film is rectangle and the film is clamped at four sides as illustrated in Fig. 2. Denote the 2-dimensional position of the film as (x, y) and the displacement of the film at time t and the position (x, y) as $u(t, x, y)$. Physical parameters of the film are set as ρ: a energy density of the film, E: tension of the film. Then, the continuous Lagrangian density of the film is given by

$$L^c = \frac{1}{2}\rho u_t^2 - \frac{1}{2}E(u_x^2 + u_y^2). \tag{13}$$

Note that the continuous Lagrangian density (13) contains through first-order partial derivative u_t, u_x, u_y.

Discretization setting is the same as the one explained in the previous section. From (13), we have the discrete Lagrangian density of the film as

$$L_{k,l,m}^d = \frac{hd_xd_y}{2}\left\{\rho\left(\frac{U_{k+1,l,m} - U_{k,l,m}}{h}\right)^2 - E\left(\frac{U_{k,l+1,m} - U_{k,l,m}}{d_x}\right)^2 - E\left(\frac{U_{k,l,m+1} - U_{k,l,m}}{d_y}\right)^2\right\}, \tag{14}$$

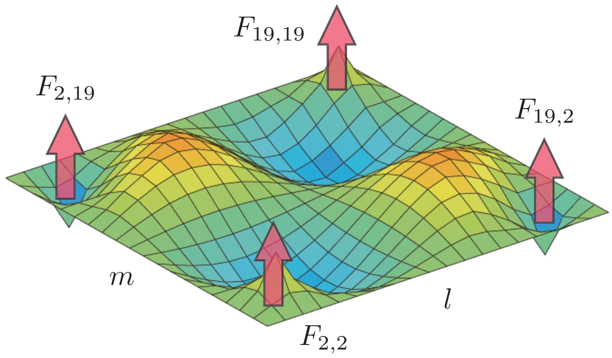

Figure 2: Film

and hence from (10) we obtain the discrete Euler-Lagrange equation of the film as

$$\begin{aligned} &-\frac{\rho}{h^2}(U_{k+1,l,m} + U_{k-1,l,m}) + \frac{E^2}{d_x^2}(U_{k,l+1,m} + U_{k,l-1,m}) \\ &+ \frac{E^2}{d_y^2}(U_{k,l,m+1} + U_{k,l,m-1}) \\ &+ 2\left(\frac{\rho}{h^2} - \frac{E}{d_x^2} - \frac{E}{d_y^2}\right)U_{k,l,m} = F_{k,l,m}. \end{aligned} \tag{15}$$

We see that (15) contains 7 displacement variables $U_{k-1,l,m}$, $U_{k,l-1,m}$, $U_{k,l,m-1}$, $U_{k,l,m}$, $U_{k,l,m+1}$, $U_{k,l+1,m}$, $U_{k+1,l,m}$. In computation of numerical solutions, a numerical stability condition called “a von Neumann condition” is quite important [9]. The next proposition gives a von Neumann condition for the discrete Euler-Lagrange equation of the film. (due to limitations of space, the proof is omitted.)

Proposition 1 : A von Neumann condition such that the discrete Euler-Lagrange equation of the film (15) is numerically stable is given by

$$0 < \frac{E}{\rho}h^2\left(\frac{1}{d_x^2} + \frac{1}{d_y^2}\right) \le 1. \tag{16}$$

Then, a numerical simulation is performed by the proposed control method. We assume that the number of actuators is 4 and they are installed at four corners of the film as illustrated in Fig. 2. The parameters are set as the physical parameters: $\rho = 1$, $E = 1$, the sampling intervals: $h = 0.01$, $d_x = 0.1$, $d_y = 0.1$, the total steps: $K = 50$, $L = 20$, $M = 20$, the set of actuator indices: $\Delta = \{(2,2), (2,19), (19,2), (19,19)\}$, the start time step of stabilization: $\kappa = 45$, the weight parameters of evaluation function: $a = 1$, $b = 3000$, $c = 1$. Note that these parameters satisfy the von Neumann condition (16).

Fig. 3 shows the simulation result on a 3D plot of the displacements of the film $U_{k,l,m}$, From this figure, it can be confirmed that all the displacements of the film converge to 0, and hence vibration suppression control is achieved. It is also possible to stabilize the film at earlier time step by tuning the parameters in the evaluation function: a, b, c.

5. Conclusions

This study has developed discrete mechanics for 2-dimensional distributed parameter mechanical systems and a new control method by blending of discrete mechanics and nonlinear optimization. A numerical simulation for a film has shown that vibration of the film is suppressed by control inputs, and then the whole of the film is stabilized by the proposed method.

The future work includes the next topics; theoretical analysis on discrete Euler-Lagrange equations and development of feedback-type controllers.

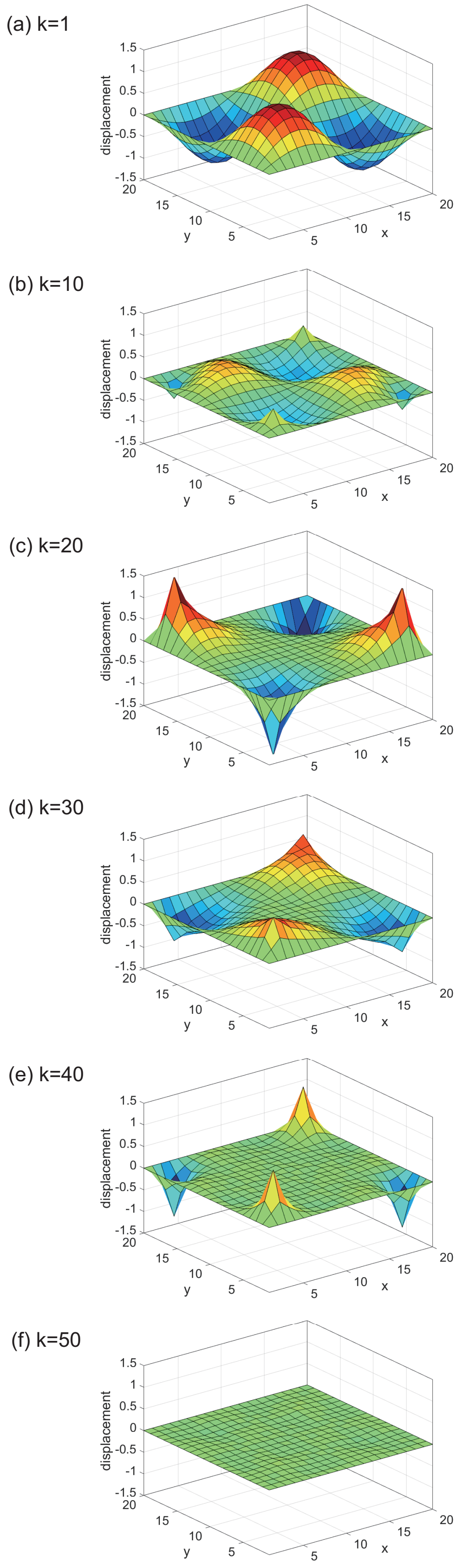

Figure 3: Snapshot of Film

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