2016 International Symposium on Nonlinear Theory and Its Applications,
NOLTA2016, Yugawara, Japan, November 27th-30th, 2016

Homoclinic bifurcations in a piece-wise constant neuron model

Chiaki Matsuda† and Hiroyuki Torikai‡

†Department of Computer Science, Kyoto Sangyo University
Motoyama, Kamigamo, Kita-Ku, Kyoto 603-8555, Japan
g1345297@cse.kyoto-su.ac.jp, torikai@cse.kyoto-su.ac.jp

Abstract—In this paper, a modified piece-wise constant neuron model is presented. It is shown that the circuit exhibits a homoclinic bifurcation known as a blue-sky catastrophe, which is also observed in a conductance-based neuron model. Also, a bifurcation diagram is obtained based on numerical experiments. Using the bifurcation diagram, occurrence mechanism of the bifurcation is explained.

1. Introduction

A piece-wise constant (PWC) neuron model has a piece-wise constant vector field and has been design to mimic bifurcations of neurons [1]-[3]. Advantages of the PWC neuron model include the following points: (a) it can be implemented by using standard circuit elements (highly specialized device technology is not needed), (b) its dynamics can be analyzed by using bifurcation analysis techniques for piece-wise constant vector fields, and (c) its vector field is relatively robust against parameter mismatch. It has been shown that the PWC neuron model can exhibit bifurcations of resting states (stable equilibrium points) such as supercritical and subcritical Hopf bifurcations, saddle-node on and off invariant circle bifurcations, and saddle homoclinic bifurcation [1]-[3], where these bifurcations have been analyzed extensively. It should be emphasized that these bifurcations are underlying mechanisms of neuron-like nonlinear responses of the PWC neuron model such as class 1 excitability, class 2 excitability, IF-curve with hysteresis, and IF-curve without hysteresis. In addition, recently, it was shown that the PWC neuron model can also exhibit bifurcations of spiking states (stable periodic orbits) such as a homoclinic bifurcation of spiking states know as a blue-sky catastrophe [1]-[3]. However, analysis of the homoclinic bifurcation of the PWC neuron model has been insufficient so far.

In this paper, a modified PWC neuron model is presented. It is shown that the modification leads to occurrence of a blue-sky catastrophe, which is more similar to that of a Hodgkin-Huxley type conductance-based neuron model [4] than our previous PWC neuron model [3]. Then, a bifurcation diagram is obtained based on numerical simulations. Using the bifurcation diagram, occurrence mechanism of the blue-sky catastrophe is explained. It should be emphasized that the analysis result in this paper will be a preliminary result to develop theoretical analysis methods of homoclinic bifurcations of the PWC neuron model and to develop bifurcation-based design methods of the PWC neuron model.

2. Blue-sky catastrophe

Let us begin with a Hodgkin-Huxley type conductance-based neuron model is described by the following equation [5]: $C\frac{dv}{dt} = -\bar{g}_{K2}m_{K2}^2(V-E_K) - g_l(V-E_l) - \bar{g}_{Na}h_{Na}(V-E_{Na})f(-150, 0.0305, V)^3 - I_{pol}$, $\tau_{K2}\frac{dm_{K2}}{dt} = f(-83, 0.018 + V_{K2}^S, V) - m_{K2}$, $\tau_{Na}\frac{dh_{Na}}{dt} = f(500, 0.0325, V) - h_{Na}$, where v is a membrane potential and m_{K2} and h_{Na} are gate variables. Also, f is a nonlinear function given by $f(a, b, V) = (1 + e^{a(V+b)})^{-1}$. It is known that the above model exhibits a homoclinic bifurcation known as the blue-sky catastrophe. In order to investigate the essential structure of the blue-sky catastrophe, the following normal form is sometimes used [4].

$$\begin{aligned}dw/dt &= x(2+\mu-b(x^2+y^2)) + z^2 + y^2 + 2y,\\ dy/dt &= -z^3 - (y+1)(z^2+y^2+2y) - 4x + \mu y,\\ dz/dt &= z^2(y+1) + x^2 - \epsilon.\end{aligned}$$

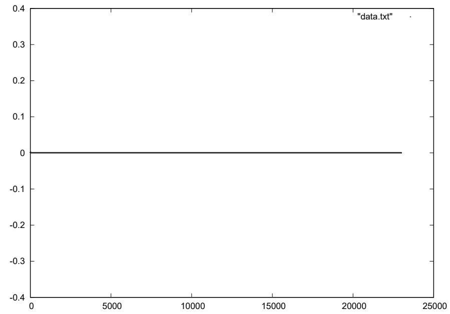
Fig. 1 shows time waveforms of the normal form of the blue-sky catastrophe. Fig. 1(a), the variable v is oscillating with very high frequency. Fig. 1(b), the parameter μ is slightly changed and the variable x suddenly starts to burst. Fig. 1(c), the parameter μ is further changed and the inter-burst-interval of the variable x becomes shorter. Since the complicated bursting orbit (catastrophe) in Fig. 1(b) seems to appear suddenly from somewhere (blue-sky), which is not related to the periodic orbit in Fig. 1(a), this bifurcation phenomenon is called a blue-sky catastrophe.

3. Modified PWC Neuron Model

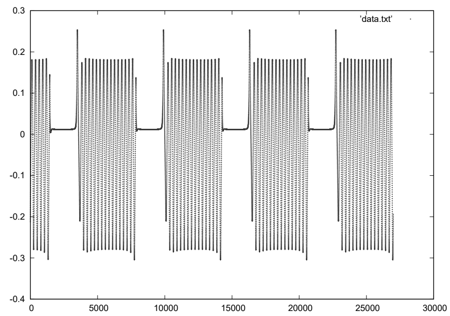
Fig. 2(a) shows a modified piece-wise constant (PWC) neuron model. The capacitor voltages v and u correspond to a membrane potential and recovery variable of a neuron model, respectively. As shown in Fig. 2(b)-(d), the current sources I_v, I_u, and I_{MH} are voltage-controlled and have the following PWC characteristics, i.e., step-function-like characteristics.

$$I_v(v_e) = \begin{cases} +I_v^+ & \text{if } v_e \geq 0,\\ -I_v^- & \text{if } v_e < 0,\end{cases}$$

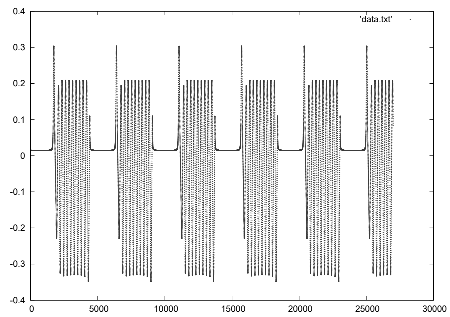
$$I_u(u_e) = \begin{cases} +I_u^+ & \text{if } u_e \geq 0,\\ -I_u^- & \text{if } u_e < 0,\end{cases}$$



(a)



(b)



(c)

Figure 1: Blue-sky catastrophe of its normal form [4]. $b = 10$. (a) $\mu = 0$, $\epsilon = 0$. (b) $\mu = 0.03$, $\epsilon = 0.02$. (c) $\mu = 0.4$, $\epsilon = 0.02$

$$I_{MH}(v, u) = \begin{cases} -I_{MH}^{-} & \text{if } v \geq V_B(u) \\ & \text{and } U_{MH}^{-} \leq u \leq U_{MH}^{+}, \\ +I_{MH}^{+} & \text{otherwise.} \end{cases}$$

Due to the PWC characteristics of the current sources, the PWC neuron model has a piece-wise constant vector field. The switch is voltage-controlled and realizes a firing reset of the membrane potential v to a voltage-controlled reset level V_B, which has the following characteristics as shown in Fig. 1(e).

$$V_B(u) = \begin{cases} V_S & \text{if } u \geq Q, \\ \alpha u + \beta & \text{if } u < Q, \end{cases}$$

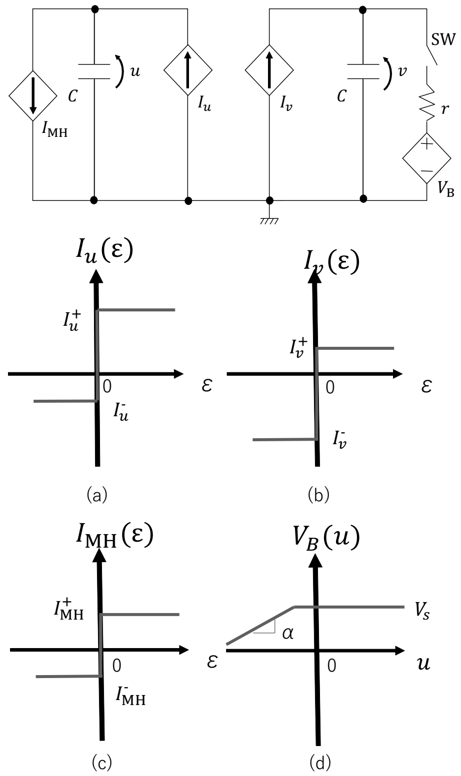


Figure 2: (a) Modified PWC neuron model. (b)-(c) Characteristics of piece-wise constant (PWC) voltage-controlled current sources I_v, I_u and I_{MH}. (d) Characteristics of voltage-controlled voltage source V_B.

Then, the dynamics of the PWC neuron model is described by the following equations.

If $v(t) < V_A$ (i.e., the switch SW is opened), then

$$C\frac{dv}{dt} = I_v(|v| + V_{in} - u),$$

$$C\frac{du}{dt} = I_u(av - u) + I_{MH}(v, u).$$

If $v(t) = V_A$ (i.e., the switch SW is closed), then

$$v(t^+) = V_B(t).$$

where $t^+ = \lim_{\delta \to 0} t + \delta$, $\delta > 0$. Fig. 3 shows typical time waveforms of the membrane potential v and the recovery variable u of the PWC neuron model. Fig. 4 shows a vector field of the PWC neuron model. As shown in Fig. 4, the characteristics of the voltage controlled voltage source V_B is controlled by the recovery variable u. Due to this voltage controlled voltage source V_B, the time waveform of the membrane potential v becomes more similar to that of the conductance-based neuron model compared to our previous model [3].

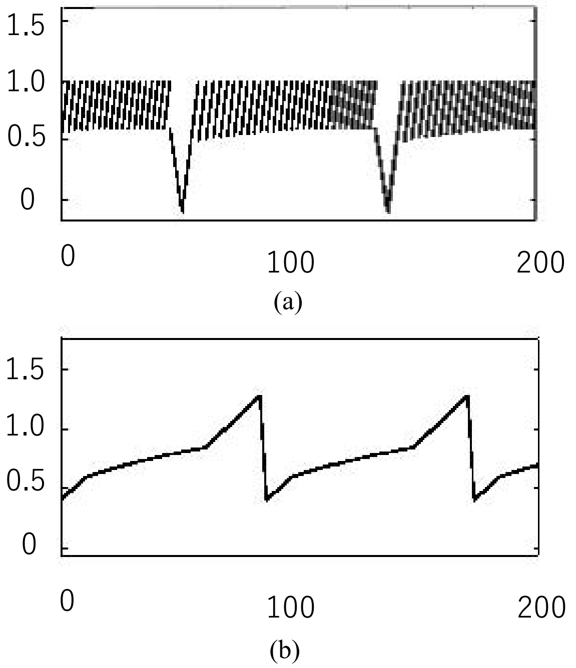


Figure 3: Typical time waveforms of the (a) membrane potential *v* and (b) the recovery variable *u* of the PWC neuron model.

4. Analysis of Homoclinic Bifurcation

Fig. 5 shows typical time waveforms and corresponding phase plane trajectories of the PWC neuron model. In Figs. 5(a) and (a'), the PWC neuron model has a stable periodic orbit (SB) corresponding to a spiking orbit with high frequency and an unstable periodic orbit (UB), where UB is shown by a dashed orbit in (a') and is not shown in (a). In Figs. 5(b) and (b'), the parameter I_{MH}^{+} is slightly changed from (a) and (a'). In this case, the SB and UB merged and disappeared. Instead of the disappeared orbits, a complicated orbit suddenly appeared as shown in 5(b) and (b'). In Figs. 5(c) and (c'), the parameter I_{MH}^{+} is further changed from (b) and (b'). In this case, the inter-burst-interval of the membrane potential *v* becomes shorter.. The mechanism of the above change of phenomena is conceptually same as the occurrence mechanism of the blue-sky catastrophe [4]. In order to analyze the blue-sky catastrophe, a two-parameter bifurcation diagram in Fig. 6 is obtained by numerical experiments. The points (a)-(c) in the diagram correspond to Figs. 5(a) and (a')-5(c) and (c'), respectively. In the region (A), the PWC neuron model has an SB and a UB. In the region (B), the PWC neuron model has a complicated bursting orbit. On the border of the regions (A) and (B), the SB and the UB merge and disappear and the complicated bursting orbit appears. At the point (c), properties of unstable manifold changes from those in (A) and (B). Note that this bifurcation diagram will be a preliminary ingredient to develop a systematic design procedure of the PWC neuron model to exhibit homoclinic bifurcations.

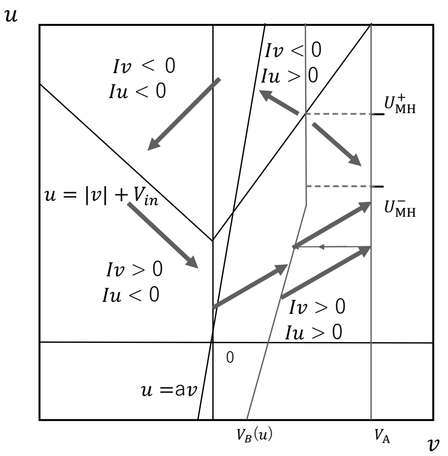


Figure 4: PWC vector field of the PWC neuron model.

5. Conclusions

In this paper, the modified PWC neuron model was investigated. It was shown that the model exhibits the blue-sky catastrophe, which is also observed in a conductance-based neuron model. Also, a bifurcation diagram was obtained based on the numerical experiments. Using the diagram, the occurrence mechanism of the blue-sky catastrophe was explained. Future problems include: (a) theoretical analysis of the blue-sky catastrophe and other homoclinic bifurcations in the PWC neuron model, (b) development of a network of the PWC neuron models, and (c) their CMOS implementations. This work was partially supported by JSPS KAKENHI Grant Number 15K00352.

References

[1] Yutaro Yamashita and Hiroyuki Torikai, Theoretical Analysis for Efficient Design of a Piece-wise Constant Spiking Neuron Model, IEEE Trans. CAS-II Vol. 60, No. 1, pp. 54 - 58, 2014.

[2] Yutaro Yamashita and Hiroyuki Torikai, A Novel PWC Spiking Neuron Model: Neuron-Like Bifurcation Scenarios and Responses, IEEE Trans. CAS-I, Vol. 59, No. 11, pp. 2678 - 2691, 2012.

[3] Souhei Takeda and Hiroyuki Torikai, Design of Homoclinic Bifurcation in PWC Neuron Model, Proc. NOLTA, pp. 345-348, 2015.

[4] Yuri A. Kuznetsov, Elements of Applied Bifurcation Theory, Springer, 2004.

[5] Andrey Shilnikov and Gennady Cymbalyuk, Transition between Tonic Spiking and Bursting in a Neuron Model via the Blue-Sky Catastrophe, Phys. Rev. Lett., Vol. 94, 048101, 2005.

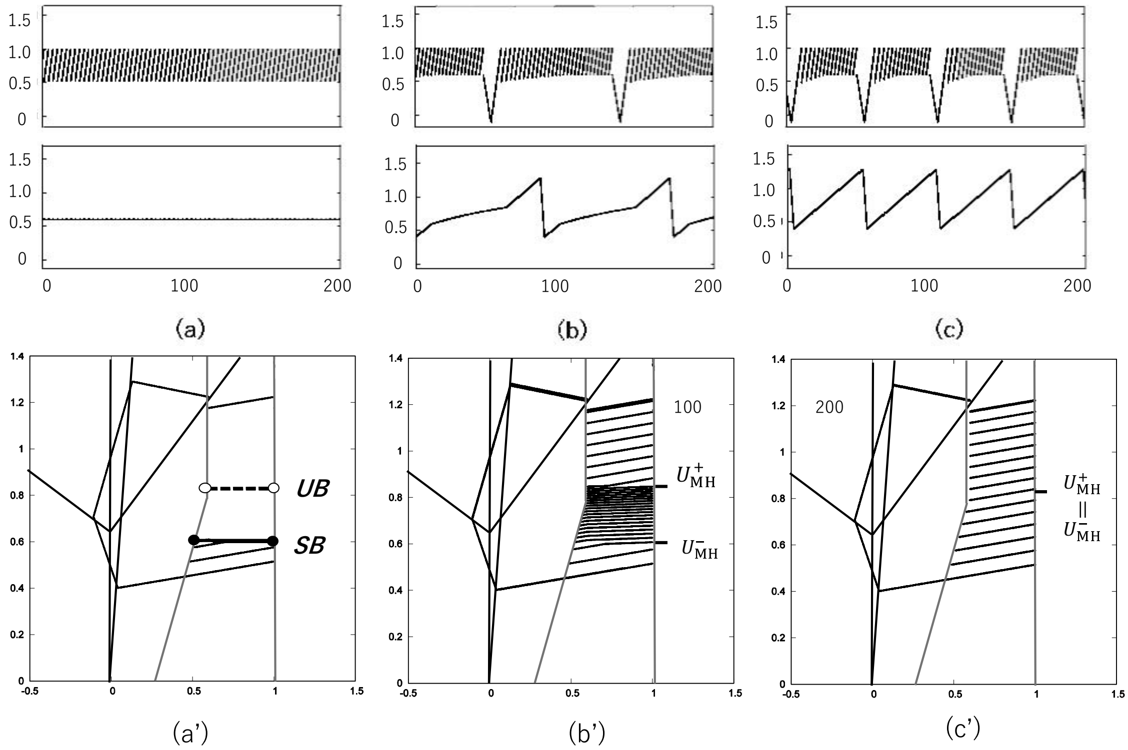


Figure 5: Bifurcation of homoclinic periodic orbits. (a)-(c) show time waveforms and (a')-(c') show corresponding phase plane orbits. SB and UB stand for "stable periodic orbit" and "unstable periodic orbit." V_B=0.6, V_A=1.0, I_u^+=0.019, I_u^-=0.33, I_v^+=0.16, I_v^-=0.135, a=10.0, V_{in}=0.6. (a) and (b) Coexistence of SB and UB. I_{MH}^+=0.07, I_{MH}^-=0, U_{MH}^+=0.85, $U_{MH}^- = 0.6$. (b) and (b') Disappearance of the SB and UB and appearance of a complicated bursting orbit. I_{MH}^+=0.015 and $U_{MH}^- = 0.6$. (c) and (c') Another disappearance of the SB and UB and appearance of a complicated bursting orbit. I_{MH}^+=0.07 and $U_{MH}^- = 0.85$.

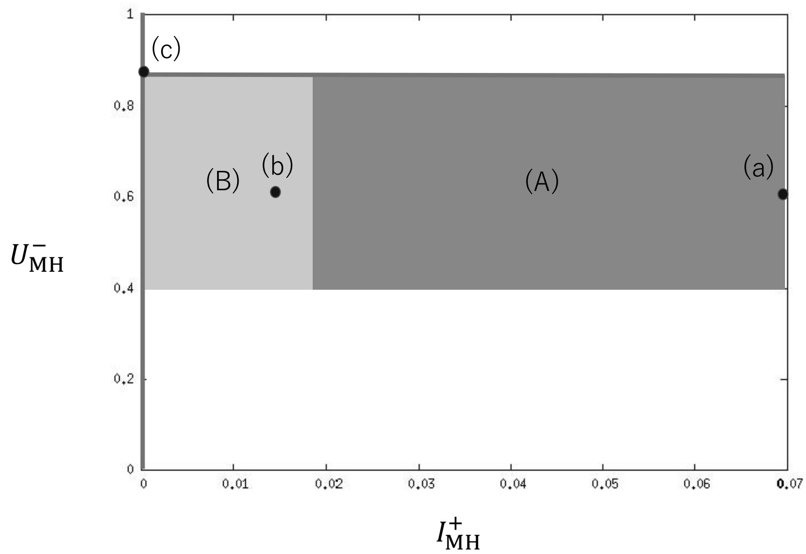


Figure 6: Bifurcation diagrams obtained by numerical experiments. The points (a)-(c) in the diagram correspond to Figs. 5(a) and (a')-5(c) and (c'), respectively. In region (A), the PWC neuron model has an SB and a UB. In region (B), the PWC neuron model has a complicated bursting orbit. On the border of the regions (A) and (B), the SB and the UB merge and disappear and the complicated bursting orbit appears.