2016 International Symposium on Nonlinear Theory and Its Applications,
NOLTA2016, Yugawara, Japan, November 27th-30th, 2016

Fredholm Determinants of Generalized β-Transformations and MSE Estimates of Corresponding AD-Converters

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Abstract—We consider an Analog-to-Digital (AD) encoder defined by generalized β-transformation. Such encoders are called β-encoders. In this article, we show that the mean squared error of a β-encoder can be estimated by analyzing the zeros of the Fredholm determinant of the transformation. We give an example of a rigorous upper bound of the MSE by this technique, together with the numerical verification method.

1. Introduction

A β-encoder is an Analog-to-Digital (AD) encoder based on the β-expansion. Compared to the conventional encoders based on the binary expansion, it is proposed that they show self-correcting property ([1]) and accordingly are advantageous for designing less energy consumption and smaller encoders. Experiments witness these advantages (see for instance [2]). While β-encoders have nice properties from the viewpoint of engineering, their mathematical treatment contains a lot of challenging problems.

In this paper, we consider the problem of finding the upper bound of mean squared error (MSE) of such AD-converters. An analog input is encoded to the digital output through β-encoders. In practice, we can only maintain finite number of digits thus there is loss of information. For optimizing the design of the encoder, it is important to establish better upper bound of MSE.

This problem is translated to estimate a certain integral of a function given by the iteration of transfer curve of the AD-encoder under consideration. This is is an easy problem for conventional AD-converters since the points of discontinuity distribute evenly over the interval. On the other hand, this is a very complicated problem for β-encoders: In general, the distribution of the points of discontinuity are scattered over the interval unevenly. Furthermore, since the transfer curve has positive Lyapunov exponent $\log\beta$, the iteration exhibits the sensitive dependence on the initial condition as the number of iteration increases, that is, when we consider β-converter with large number of digits. As a result, the behavior of the MSE shows quite a complicated behavior when the parameter value β varies.

A simple MSE upper bound for β-encoders is obtained in [3], by means of change of the variable argument and approximation of integrals by a Markov process of finite states. The result is rigorous. However, since the proof is done by approximation method, as the number of digits increases the amount of calculation required for obtaining the inequality increases. Consequently, the conclusion we could obtain was limited. We gave MSE only for the case for limited number of digits.

In this article, we propose a new method for estimating the MSE of β-encoders. It is based on the analysis of the eigenvalues of the Perron-Frobenius operator of the transformation. Roughly speaking, in this method we consider the statistic property of infinite-states Markov process directly, that is, without approximation. The advantage of this method is that the size of calculation does not change as the number of digits increases. Indeed, by this method we can derive an upper bound of MSE valid for all (sufficiently large) digits.

To establish the inequality, we need to estimate two quantities. Namely, the absolute value of the second eigenvalue of Perron-Frobenius operator and the coefficient of the corresponding decay term. By numerical verification method, we can give a rigorous upper bound for these quantities. Because of the limitation of pages, we do not discuss much about this part in this paper.

The organization of this article is as follows: In section 2 we present basic definitions and give the precise statement of our problem. In section 3 we review the calculation of Fredholm determinant of the Perron-Frobenius operator of β-transformations. In section 4 using the result of section 3 we give an upper bound of MSE. In section 5 we present the numerical result.

2. Precise statement

2.1. Setting

Let $\beta \in (1, 2]$ and $\nu \in [1-\beta^{-1}, \beta^{-1}]$. We consider the transformation on $I = [0, 1]$ defined as follows (notice that it is slightly different from the usual β-transformation):

$$T_{\beta,\nu}(x) = T(x) = \begin{cases} \beta x & (x \le \nu) \\ \beta(x-1)+1 & (x > \nu). \end{cases}$$

For $x \in I$, we also define the i-th digit $d_i(x)$ of x by

$$d_i(x) = \begin{cases} 0 & (T^{i-1}(x) \le \nu) \\ 1 & (T^{i-1}(x) > \nu). \end{cases}$$

Then, the infinite sequence $(d_i(x)) \in \{0,1\}^{\mathbb{N}}$ gives a β-*expansion* of x, namely, we have the following equality:

$$x = (\beta - 1)^{-1} \left(\sum_{i=1}^{+\infty} \frac{d_i(x)}{\beta^i} \right). \tag{1}$$

The coefficient $(\beta - 1)^{-1}$ is the normalization constant. The sequence $(d_i(x))$ is the digital encoding of an input x and by (Eq: 1) we can recover the input from the infinite $\{0,1\}$-sequence.

In an ideal situation where all the (infinitely many) digits are available, the encoding and decoding process does not bring any loss of information. However, in the real world only finitely many digits are available. By $L \in \mathbb{N}$ we denote the number of digits available. Then for an input x, to recover its original value from the quantization $(d_i(x))$, instead of (Eq. 1) the resulted decoding is given by the formula below:

$$\bar{x}_L(x) = (\beta - 1)^{-1} \left(\sum_{i=1}^{L} \frac{d_i(x)}{\beta^i} \right) + \theta\beta^{-L}. \tag{2}$$

The term $\theta\beta^{-L}$ is added in order to decrease the loss of information in the average.

The aim of this paper is to give an estimate of *mean squared error* through this imperfect encoding-decoding process, that is, to give an upper bound of the integral

$$\mathrm{MSE}(\beta, \nu, L) := \int_0^1 |\bar{x}_L(x) - x|^2 dx.$$

For fixed β and ν, it is natural to guess $\mathrm{MSE}(\beta, \nu, L) = O(\beta^{-2n})$. We are interested in estimating the coefficient of right hand side.

2.2. Segments and Populations

For analyzing the MSE, we introduce an important sequence of integers called *segments*, introduced in [3]. Given $k \geq 1$, the points of discontinuity of T^k divides the interval I into sub-intervals (thus each point in such the same sub-interval has the same expansions up to k-th digits). The number of such intervals increases exponentially with respect to k hence hard to deal with. However, if we take the image of them under T^k, then we have the following:

1. every image interval appearing in T^k is contained in that of T^{k+1};
2. there are at most two new image intervals in T^{k+1} which do not appear in T^k, accordingly, the number of image intervals appears at most linearly.

We label each image interval $J_0, J_1, \ldots J_{2k-1}$ and call them segments of T^k. Notice that some of J_i may coincide or be empty. We denote the population (number) of J_i in T^k by $n_i^{(k)}$.

Then, by a change of variable argument, we have the following equality:

$$\mathrm{MSE}(\beta, \nu, L) = \beta^{-3L} \sum_{i=0}^{2L-1} n_i^{(L)} I_i,$$

where (I_i) are real numbers given as follows: We put $J_i = [l_i, r_i]$. Then, $I_i = \frac{1}{3}\left[(r_i - \theta)^3 - (l_i - \theta)^3\right]$. Thus if we have some estimate about the behavior of $n_i^{(L)}$ then we obtain an upper bound of MSE.

Intuitively, we may guess that for fixed β and ν, MSE has order β^{-2L}. Thus in the following we are interested in estimating the following constant:

$$K_{\beta,L} := \sum_{i=0}^{2L-1} \frac{n_i^{(L)}}{\beta^L} I_i, \tag{3}$$

3. Perron-Frobenius operator and segments

3.1. Perron-Frobenius operator

For a piecewise C^1 transformation S of an interval I, we can define the Perron-Frobenius operator P acting on $L^1(I)$ as the (extension of the) adjoint operator of the Koopman operator $f(x) \mapsto f(S(x))$ with respect to the L^2 inner product. We are interested in calculating $P(\mathbb{1}_J)$ where P is the Perron-Frobenius operator for T introduced in the previous subsection and $\mathbb{1}_J$ is the characteristic function of an interval $J \subset I$. For such functions, we have

$$P(\mathbb{1}_J) = (1/\beta)(\mathbb{1}_{T(J_-)} + \mathbb{1}_{T(J_+)}),$$

where $J_- = J \cap [0, \nu]$ and $J_+ = J \cap [\nu, 1]$.

Notice that by definition we have

$$P^L(\mathbb{1}_I) = \beta^{-L} \sum_{i=0}^{2L-1} n_i^{(L)} \mathbb{1}_{J_i}.$$

Thus, the analysis of the MSE is reduced to the study of corresponding Perron-Frobenius operator. In the following, we investigate the behavior of the sequence of functions $(P^L(\mathbb{1}_I))$.

3.2. Generating function

To analyze $(P^L(\mathbb{1}_I))$, we introduce a nice tool to analyze sequences satisfying recursive relations called *generating function*. Consider the following formal power series of functions

$$s^I(z) := \sum_{i=0}^{\infty} (P^i(\mathbb{1}_I)) z^i,$$

where z is a formal variable. Notice that formally we have $s^I(z) = (1 - zP)^{-1}(\mathbb{1}_I)$. This suggests that the solution of the equation $s^I(z) = 0$ is the reciprocal of the eigenvalue of the operator P.

3.3. Fredholm determinant

Using the recursive relation of the sequence $(P^L(\mathbb{1}_I))$, we can derive a closed formula of $s^I(z)$. This is done for the greedy β-transformation by Ito and Takahashi ([4]). For general case, this is done by Mori ([5]). For simplicity, in the following we assume that $\nu = \beta^{-1}$, that is, the left branch of T covers the whole I. In this case, in each iteration of T there is at most only one non-empty segment. Thus we forget the a priori empty segments and denote the non-empty ones as $J_0, \dots J_L$. We choose θ to be the middle point of J_1, that is, $\theta = (3-\beta)/2$. In this setting, $s^I(z)$ is given by the following formula:

$$s^I(z) = \frac{1}{(1-z/\beta)(1-E(z))}\left(\sum_{i=1}^{\infty}(z/\beta)^i \mathbb{1}_{s_i}\right), \tag{4}$$

where

$$E(z) = \sum_{i=1}^{\infty} \phi(i-1)\left(\frac{z}{\beta}\right)^i, \quad \phi(i) = \begin{cases} 0 & (T^{i+1}(1/\beta) \geq 1/\beta), \\ 1 & (\text{otherwise}). \end{cases}$$

Notice that, while the domain of convergence of $s^I(z)$ was initially $|z| < 1$, $s^I(z)$ converges for $|z| < \beta$ in the new formula. Thus we have obtained an analytic continuation of $s^I(z)$.

By taking the Taylor expansion of $s^I(z)$, we can extract some information about $n_i^{(L)}$. We put

$$\frac{1}{(1-z/\beta)(1-E(z))} = \sum_{i=0}^{\infty} w_i z^i. \tag{5}$$

By expanding (Eq:4) and comparing the coefficients, we obtain

$$n_i^{(L)} = w_{L-i}\cdot\beta^{L-i}. \tag{6}$$

Thus, in order to obtain the estimate of MSE, we need to know the behavior of the sequence (w_n).

3.4. Taylor expansion of coefficient function and MSE

Let us estimate the coefficients (w_i). In our setting, we can prove that $z = 1$ is a simple root of $1 - E(z) = 0$ (see Mori for example).

Thus we have the following factorization:

$$1 - E(z) = (1-z)R(z).$$

where $R(z)$ is a holomorphic function on $|z| < \beta$. By a simple calculation together with the fact that $(\phi(i))$ is related to the β-expansion of $1/\beta$, we have

$$R(z) = \sum_{i=0}^{\infty} \frac{1 - f^{i+1}(1/\beta)}{(\beta-1)\beta^i} z^i.$$

Thus, (Eq:5) is equal to

$$\frac{1}{(1-z/\beta)}\left[\frac{1-rR(z)}{(1-z)R(z)}\right] + \frac{r}{(1-z/\beta)(1-z)},$$

where $r = 1/R(1)$.

The Taylor expansion of these terms provides us with information of (w_n). For instance, the coefficient of z^n from the second term (we denote it by u_n) is

$$u_n = \frac{r}{\beta-1}(\beta - \beta^{-n}).$$

Let us estimate the coefficient of the first term.

3.5. Second eigenvalue and estimate of coefficient

In order to obtain the estimate of contribution of the first term, we use contour integrals. We denote the coefficient of z^n of the first function by v_n. Notice that the function in the integral has no pole in $|z| \leq \mu$ for every $\mu < \eta$, where η is the absolute value of zero of $R(z) = 0$ with smallest absolute value. Thus we have the following equality.

$$v_n = \frac{1}{2\pi i}\int_{z=\mu e^{i\theta}} \frac{1-rR(z)}{(1-z)(1-z/\beta)R(z)}\frac{1}{z^{n+1}}d\theta.$$

Thus if μ can be chosen greater than 1, then we know that v_n decays exponentially:

$$v_n \leq \frac{J(\mu)}{2\pi}\mu^{-(n+1)}, \tag{7}$$

where

$$J(\mu) = \left|\int_{z=\mu e^{i\theta}} \frac{1-rR(z)}{(1-z)(1-z/\beta)R(z)}d\theta\right|.$$

These constants can be calculated numerically. Indeed, adopting numerical verification method, we can establish an upper bound for η and once we fix μ, then it is possible to obtain the upper bound of $J(\mu)$ (as an upper bound of the contour integral). Together with these constants with rigorous numerical verification, we can derive several rigorous upper bounds of MSEs.

4. Estimation of MSE

4.1. Calculation of population

Recall that, in order to obtain the upper bound, we only need to obtain the upper bound of the constant $K_{\beta,L}$ in (Eq:3). Using (Eq:6) and $w_n = u_n + v_n$, we have

$$K_{\beta,L} = \sum_{i=0}^{L} \frac{r}{\beta-1}\left(\beta - \beta^{-(L-i)}\right)\frac{I_i}{\beta^i} + \sum_{i=0}^{L} v_{L-i}\frac{I_i}{\beta^i}.$$

Let us consider the case where L is sufficiently large (say $L \geq 18$), since the case where L is smaller than these values are treated in the paper [3].

To obtain the estimate of $K_{\beta,L}$, we divide the sum into two parts: the part $i \leq 10$ and the part $i > 10$. We denote the former one by K_{10} and the other by K_{res}. The term K_{10} is not hard to estimate since we consider β in a small interval. Hence we calculate it directly.

The latter part is hard to estimate since it is related to the dynamics of higher iteration. Thus we only give an upper bound to it. The quantity we want to estimate is:

$$\sum_{i=11}^{L} \frac{r}{\beta-1}\frac{I_i}{\beta^{i-1}} - \sum_{i=11}^{L} \frac{r}{\beta-1}\frac{I_i}{\beta^{n}} + \sum_{i=11}^{L} v_{L-i}\frac{I_i}{\beta^{i}}.$$

For the second term, since it is negative and small we neglect it. For the first and the third term, we substitute I_i with its worst value $\tilde{I}$, that is, we assume $l_i = 0$ and $r_i = 1$. Namely, recalling that we put $\theta = (3-\beta)/2$,

$$\begin{aligned} \tilde{I} &= \frac{1}{3}\left[(1-\frac{1}{2}(3-\beta))^3 - (\frac{1}{2}(3-\beta))^3\right] \\ &= \frac{1}{3}(3\beta^2 - 12\beta + 13). \end{aligned}$$

By this substitution, for the first term we have

$$\sum_{i=11}^{L} \frac{r}{\beta-1}\frac{I_i}{\beta^{i-1}} \leq \frac{r}{\beta^{10}(\beta-1)} \sum_{i=0}^{+\infty} \frac{\tilde{I}}{\beta^{i}} = \frac{r\tilde{I}}{\beta^{9}(\beta-1)^2}.$$

For the third term, we use the upper bound obtained by the second eigenvalue. By (Eq:7), for every $11 \leq i \leq L$ we have

$$\left| v_{L-i}\frac{I_i}{\beta^i} \right| \leq \frac{J(\mu)}{2\pi\mu^{L-i+1}}\frac{\tilde{I}}{\beta^i} \leq \frac{J(\mu)\tilde{I}}{2\pi\mu}\left(\frac{\mu}{\beta}\right)^i \mu^{-L} =: K_a.$$

Accordingly, we have

$$\begin{aligned} &\left| \sum_{i=11}^{L} v_{L-i}\frac{I_i}{\beta^i} \right| \leq \frac{J(\mu)\tilde{I}}{2\pi\mu^{L+1}} \sum_{i=11}^{L} \left(\frac{\mu}{\beta}\right)^i \\ \leq\ & \frac{J(\mu)\tilde{I}}{2\pi\mu^{L+1}}\left(\frac{\mu}{\beta}\right)^{11} \sum_{i=0}^{+\infty}\left(\frac{\mu}{\beta}\right)^i = \frac{J(\mu)\tilde{I}}{2\pi\mu^{L+1}}\left(\frac{\mu}{\beta}\right)^{11} \frac{\beta}{\beta-\mu} =: K_b. \end{aligned}$$

Finally, we have $K_{\beta,L} = K_{10} + K_{\mathrm{res}} \leq K_{10} + K_a + K_b$. Notice that, compared to K_{10} the other two terms are very small. Thus K_{10} is the dominant term of $K_{\beta,L}$. We also remark that, by examining the calculation carefully, the above upper bound is valid as the upper bound for $K_{\beta,M}$ for every $M \geq L$. Thus our result provides the upper bound not only for specific L but also for every $K_{\beta,M}$ with $M \geq L$.

5. Numerical result

5.1. Second eigenvalues

For the sake of simplicity, we concentrate on the case where $\beta \in [1.83, 1.8300001]$. We believe that the same technique would provide similar results with more calculation. In this case, we can prove the following:

- For the β above, let $\hat{\mu}$ be the root of $R(z) = 0$ with the smallest eigenvalue. Then we have $|\hat{\mu}| > 1.622531$.
- Letting $\mu = 1.55$, we have $J(\mu) < 4.774380$.

5.2. The estimation of MSEs

By these results, combining the result in the previous section, we obtain the following:

- For the β above, we have $0.0449 < K_{10} + K_{\mathrm{rem}} < 0.0450$.
- The error between this value and numerically estimated $K_{\beta,20}$ is less than 1%.

6. Summary

By means of spectral analysis of Perron-Frobenius operators, we derived an upper bound for MSE of β-encoders. This upper bound is valid not only for some specific number of digits but but also for every sufficiently large digits. We believe that by this method we can establish the upper bound not only for β in a narrow range considered in this paper but also for β in the other parameter range or for encoders given by different threshold. We would like to complete such research in the other opportunities.

Acknowledgments

This work was supported by JSPS KAKENHI Grant Number 16K00333.

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