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Estimation of Beta-Value for pipelined beta encoders

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Abstract—A β-converter is an analog-to-digital (A/D) encoder, that outputs truncated sequence of β expansion of an input value $x \in [0, 1)$. β-converter has been proved to be robust to the fluctuation of the threshold value in quantizer. However, it remains an unsolved issue to give an accurate estimation of the β value in a pipeline β encoder. In this paper, we propose a new method estimating β by using β-map and the accuracy is also evaluated by numerical simulations.

1. Introduction

An A/D converter converts the continuous physical value to digital number and it is implemented in various of electronic equipment. Due to the large improvement in semiconductor microfabrication technology during recent years, A/D converters also tend to be more compact and having lower power consumption. Therefore circuit element's value and threshold voltage play important roles and it is getting difficult to make sure of the conversion accuracy. For choosing an A/D conversion architecture, it is also important to concern about its electricity consumption, accuracy, conversion rate. Nowadays, i)Nyquist rate converter and ii)over sampled converter are the two main types of A/D converters which are commonly used.

Nyquist rate converter cuts off the signal whose frequency is over W by analog filter and samples the signal in the frequency over $2W$. After that, these sampled-value are converted into binary digits by the A/D converter. Among the Nyquist rate converter, the most popular one is natural weighted binary encoder also called PCM (Pulse Code Modulation) which gains the dyadic expansion of an input value $x \in [0, 1]$. Although PCM is known to be easily calculated and achieves a precision of order $O(2^{-N})$(N is bitrate), it makes error when the threshold voltage is fluctuated. On the other hand, $\Sigma\Delta$ modulation is the typical over sampled converter. It has a self-correction property to the fluctuation of threshold by over-sampling in low quantization accuracy. $\Sigma\Delta$ modulation is robust to the electric circuit elements and this property is the one that Nyquist rate converter does not own. Because of this robustness, we prefer $\Sigma\Delta$ modulation to PCM while using imperfect quantizer, although $\Sigma\Delta$ modulation owns slow conversion rate (achieves precision of order $O(N^{-1})$).

β-encoder is a new type of Nyquist rate converter proposed by Daubechies et al. in 2002 [1]. The most important fact about the β-encoder is that it is robust to the fluctuated quantizer while achieving a precision of order $O(\beta^{-N})$. β-encoder convert input-analog-signal $x \in [0, 1)$ to digital bits by the expression

$$x = \sum_{n=1}^{\infty} b_n \beta^{-n}, \tag{1}$$

where β is a real number satisfies the inequality $1 < \beta < 2$ and $b_n \in \{0, 1\}$. β encoders have a look-up table (LUT) that memorizes the binary expansion of β^{-n} ($n = 1, 2, .., \dots N$). Such a LUT is used to convert the beta expansion coefficient $\{b_n\}$s of x to binary expressoin of x. β-encoder overcomes the disadvantage of both PCM and $\Sigma\Delta$ modulation and it owns the potential to carry out A/D conversion in both high accuracy and speed.

When we mention about an β-encoder, there are two types of them. One is cyclic type which uses only one β-encoder to provide output bits and the other is pipeline β-encoder using plurality of β-encoder to provide outputs from each encoder. In order to apply faster analog digital conversion, it is important to construct pipeline β-encoder. As we can observed from expression (1), we need to know the exact β-value to restore the input value x. We propose a new method for estimating β-value in pipeline β-encoder which uses β-map. Meanwhile, we also show the result of our numerical experiment estimating the β-value using the proposed method.

2. β-encoder

2.1. cyclic-β-encoder

A β-encoder is composed of a β-times($1 < \beta < 2$) amplifier and a quantizer with threshold value ν. β-encoder is known to be much more robust to the fluctuation of circuit elements than PCM. Moreover, β-encoder also could convert in higher rate than $\Sigma\Delta$ converter, which means it can convert in both high rate and accuracy. Figure 2 shows a block diagram of β-encoder and when $\beta = 2$ it reduces to the PCM.

In a β-encoder, an input value $x \in [0, 1)$ can be expanded into $x = (\beta - 1)\sum_{n=1}^{n=\infty} b_n \beta^{-n}, b_n \in \{0, 1\}$. The expansion coefficients $\{b_n\}$ can be obtained as follows: we define a β-expansion map as

$$C_\beta(x) = \begin{cases} \beta x, & x < \nu/\beta, \\ \beta x + 1 - \beta, & x \geq \nu/\beta. \end{cases} \tag{2}$$

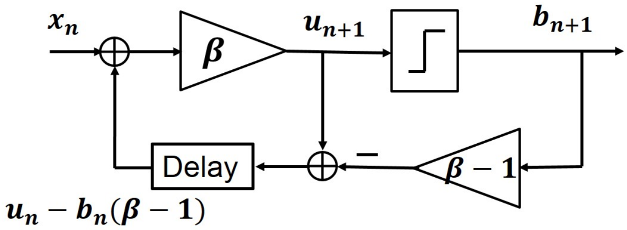

Figure 1: cyclic-β-encoder($x_0 = x$, $x_n = 0 (n > 0)$)

where $\nu \in (\beta - 1,\ 1)$ is the threshold of the quantizer. Define a quantizer as

$$Q_\nu(x) = \begin{cases} 0, & x < \nu, \\ 1, & x \geq \nu. \end{cases} \quad (3)$$

Then b_n is given by following expressions:

$$\begin{cases} u_1 = \beta x, b_1 = Q_\nu(u_1), & n = 0 \\ u_{n+1} = \beta(u_n - b_n(\beta - 1)), b_{n+1} = Q_\nu(u_{n+1}), & n > 0 \end{cases} \quad (4)$$

When we use a quantizer with error ϵ in the threshold value ν, we can still carry out A/D conversion in a precision of order $O(\beta^{-N})$ if the threshold value remains in $(\beta - 1, 1)$, which means $(\nu \pm \epsilon) \in (\beta - 1, 1)$. This characteristic is the one that PCM does not own, and so that β-encoder is said to be robust to the fluctuated quantizer.

2.2. β-estimation using Daubechies et al.'s method

In this section, a β-estimation method proposed by Daubechies et al. [2] is explained. It remains an important issue to know the exact value of β in order to conduct high accuracy A/D conversion using β encoder. However, we cannot make exact β-converter because of the fluctuation of the circuit elements. Therefore it remains an important issue to estimate β-value in high accuracy after choosing a β times amplifier with errors (Although it seems there are two β-value remain estimating in Fig. 1, the β-value in two amplifier gain same value while using MDAC circuit [5]. So only one β-value remain estimating). In terms of Daubechies et al.'s β estimating method, two input values $x \in (0, 1)$ and $1 - x \in (0, 1)$ are used to produce L bits of β-outputs $\{b_i\}_{i=1}^L$ and $\{c_i\}_{i=1}^L$. Defining $\gamma = \beta^{-1}$, $C = 1 + \nu + \epsilon$, $k_0 = \log(\frac{1-\gamma}{2}) / \log\gamma$, $C' = \max\{2C, 2C/(k_0\gamma^{(k_0-1)})\}$, ϵ the error of threshold value ν, then quantization error will satisfy the inequality

$$0 \leq x - (1/\gamma - 1)\sum_{i=1}^{L} b_i\gamma^i \leq C'\gamma^L, \quad (5)$$

where $\gamma = \beta^{-1}$. Meanwhile the quantization error of input value $1 - x$ also satisfies

$$0 \leq 1 - x - (1/\gamma - 1)\sum_{i=1}^{L} c_i\gamma^i \leq C'\gamma^L. \quad (6)$$

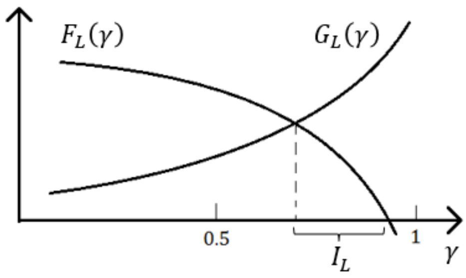

Figure 2: $F_L(\gamma)$ and $G_L(\gamma)$

From the Eqs. (5) and (6), we have

$$0 \leq F_L(\gamma) \leq G_L(\gamma), \quad (7)$$

where $F_L(\gamma) = 1 - (\frac{1}{\gamma} - 1)\sum_{i=1}^{L}(b_i + c_i)\gamma^i$ and $G_L(\gamma) = 2C'\gamma^L$.

It can be easily verified that $F_L(\gamma)$ is a monotone decreasing function of γ and that $G_L(\gamma)$ is a monotone increasing function of γ. According to these characteristics of $F_L(\gamma)$ and $G_L(\gamma)$, we can draw a graph in Fig. 2. From the inequality $0 \leq F_L(\gamma) \leq G_L(\gamma)$, we know that the true γ is limited in the region I_L illustrated in Fig. 2. Therefore when L is large enough, we could figure out the estimation value of γ expressed in $\hat{\gamma}$ by finding the solution satisfies inequality(7) with Newton's method or bisection method. After that, according to expression $\gamma = \beta^{-1}$, estimated β value $\hat{\beta}$ is calculated. In previous Oda's research [3], they have also discovered a method finding the solution of inequality(7) by gradually enlarging L. This method has a benefit that total number of calculation is decreased.The $\hat{\beta}^{-n}$s are memorized in a LUT. Let the range of β be $[\beta_{\min}, \beta_{\max}]$, which are divided into subintervals with equal width $\Delta\beta$. The LUT memorizes $\hat{\beta}^{-n}$s for all candidates $\hat{\beta} = \beta_{\min} + j\Delta\beta$ $j = 0, 1, 2....$

2.3. Pipeline β-encoder

Cyclic model and pipeline model are the two main models of the circuit structure when using Nyquist rate converter. If we assume getting L bits of output bits, the former one uses only one quantizer for L times, however, the latter one uses L pieces of quantizer to get the output bits. Consequently, the pipeline-model's circuit area is L times larger than the cyclic-model's but it could convert the signal in L times higher rate than the cyclic-model. We show the pipeline β-encoder in Fig. 3. For a pipeline β-encoder, let β_i be the amplification factor of i-th β-encoder (see Fig. 3). Then, we have

$$x_i = \beta_i x_{i-1} - (\beta_i - 1)b_i \quad (i = 1, 2, 3, ..., L) \quad (8)$$

By using β-expansion in each stage, we can also obtain the following expression:

$$x_L = \beta_L(\beta_{L-1}(\cdots(\beta_1 x_0 - (\beta_1 - 1)b_1) - (\beta_2 - 1)b_2)$$

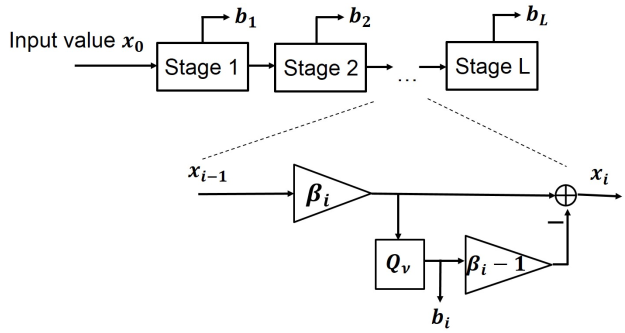

Figure 3: Pipeline β encoder

$$
\begin{aligned}
&\cdots) - (\beta_L - 1)b_L \\
= &\prod_{i=1}^{L} \beta_i x_0 - \sum_{i=1}^{L} (\beta_i - 1) b_i \prod_{j=i+1}^{L} \beta_j. \qquad (9)
\end{aligned}
$$

From this expression, we could get the input value x_0 as the following expression since x_L cannot be known.

$$
x_0 = \sum_{i=1}^{L} (\beta_i - 1) b_i \prod_{j=1}^{i} \frac{1}{\beta_j} + x_L \prod_{i=1}^{L} \frac{1}{\beta_i}. \qquad (10)
$$

Therefore, in order to reconstruct the input-value x_0 precisely from b_i, it is necessary to know the exact value of β_i.

3. β-estimation for pipeline-β encoder

In section 2.2, we discussed about β-estimation method in cyclic β-encoder. However, in the case of pipeline β-encoder, we need to estimate plural β-value. While using Daubechies's method estimating β-value, we can only achieve one inequality meaning impossible to estimate plural β. So that we need to find out a new method for estimating the β-value in pipeline β-encoder.

In this section, we explain our proposed method for estimating β-value for pipeline-β encoder. In the following arguments, we suppose that:

- $\beta_1, \beta_2, ..., \beta_L$ are unknown (we could only know the range $[\beta_{min}, \beta_{max}]$)
- The dispersion of β-value tends to be larger in the later stage, however in this paper we supposed the dispersion does not differ in each stage.
- We can obtain output bits $b_1, ..., b_L$
- Input value x_0 cannot be made accurately and we also cannot get $x_1, ..., x_{L-1}, x_L$ directly

We update the β-value from the previously estimated value to a new one by the steps bellow:

1 We get as much input value x_0 as possible (suppose K samples of x_0). $x_0^{(k)}$ means the kth input value for $k = 1, 2, ..., K$.

2 Let $b_1^{(k)}...b_L^{(k)}$ represent the output bits for the initial value $x_0^{(k)}$ from β-encoder. The value $\beta_1, \dots, \beta_{L'}$ will be the target β to estimate. We give up estimating the later $(L - L')\beta_i$, and considering their value as $\bar{\beta} = (\beta_{min} + \beta_{max})/2$. (In a practical imprementation, one can use the nominal value of beta as $\bar{\beta}$. However, in this simulation, we assume that the nominal value is given by $\bar{\beta} = (\beta_{\mathrm{min}} + \beta_{\mathrm{max}})/2$)

3 We estimate the β in the order $\beta_{L'}, \beta_{L'-1}, ..., \beta_1$ (from the later one to the former one). The estimation procedure follows the steps below:

4-1 At this stage, we calculate pairs of reconstructed values $(x_{i-1}^{(k)}, x_i^{(k)})$ using output bits $b_1^{(k)}...b_L^{(k)}$ according to the expressions:

$$
\hat{x}_{i-1}^{(k)} = \sum_{n=i}^{L} [(\hat{\beta}_n - 1) b_n^{(k)} \prod_{m=i}^{n} \beta_m^{-1}] \qquad (11)
$$

$$
\hat{x}_{i}^{(k)} = \sum_{n=i+1}^{L} [(\hat{\beta}_{n-1} - 1) b_n^{(k)} \prod_{m=i+1}^{n} \beta_m^{-1}] \qquad (12)
$$

Note that $\hat{\beta}_n$, the estimated value of β_n, have been already calculated at this moment for $n = i+1, i+2, ..., L$. Although in Eq. (11) it appears $\hat{\beta}_i$ which remain unconfirmed, we suppose its value as $\bar{\beta}$. Our aim is to estimate this $\hat{\beta}_i$.

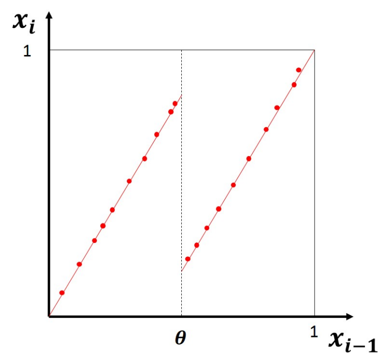

Figure 4: β-map of pipeline-β-encoder

4-2 According to Eqs. (8), (11) and (12), we have

$$
\hat{x}_i^{(k)} = \begin{cases} \beta_i \hat{x}_{i-1}^{(k)}, & \text{if } \mathrm{x_i} < \theta \\ \beta_i \hat{x}_{i-1}^{(k)} + 1 - \beta_i, & \text{if } \mathrm{x_i} \geq \theta \end{cases} \qquad (13)
$$

Then,we can figure out that the point $(\hat{x}_{i-1}^{k}, \hat{x}_i^{k})$ lies on the locus defined by the expression above. θ represents the unknown threshold value (Fig. 4). We consider the point on the most right side of left branch as $(\hat{x}_{i-1}^{(k_0)}, \hat{x}_i^{(k_0)})$ and consider the point on the most left side of the right branch as $(\hat{x}_{i-1}^{(k_1)}, \hat{x}_i^{(k_1)})$. Then, we suppose that,

$$\theta = \frac{\hat{x}_{i-1}^{(k_0)} + \hat{x}_{i-1}^{(k_1)}}{2} \tag{14}$$

$$\hat{\beta}_i = \frac{\frac{\hat{x}_i^{(k_0)}}{\theta} + \frac{1-\hat{x}_i^{(k_1)}}{1-\theta}}{2} \tag{15}$$

4-3 We update the estimated $\hat{\beta}_i$ value according to Eqs. (14) and (15). Then update $\hat{x}_{i-1}^{(k)}$ again according to Eq.(10), after that repeat step (4-2). We quit updating $\hat{\beta}_i$ after repeating a few times J. Finally, let $i \rightarrow i-1$ and go back to 4-1 until we finish estimating all the β.

4. Numerical Results

The accuracy of β-value is evaluated by computer simulation. In the simulation, we repeat the evaluation for 4 times while the number of input value x_0 varies from 10^5, 10^6, 10^7 and the number of β-encoder are $L' = 15$, $L = 30$. We suppose the ture β values are randomly selected in the range $\beta \in [1.69, 1.71]$ according to the uniform distribution and start simulation from the initial value $\overline{\beta} = 1.7$. The performances are evaluated by MSE (Mean Squared Error).

The simulation results show that the former β's MSE is getting smaller than the later one. This is because the accuracy is gradually becoming accurate from the later β to the former β. And comparing Fig. 5 and Fig. 6, the MSE is smaller as the input number K increase. From Fig. 7, we also know that when the repetition frequency increase, MSE does not decrease.

5. Conclusion

We proposed a new method estimating β-value in pipeline β encoder and conducted experiment through computer program. In our experiment, the MSE decrease comparing to the original error which implies the availability of our proposed method. We would like to implement our method in a electric circuit and verify its performance in the future work.

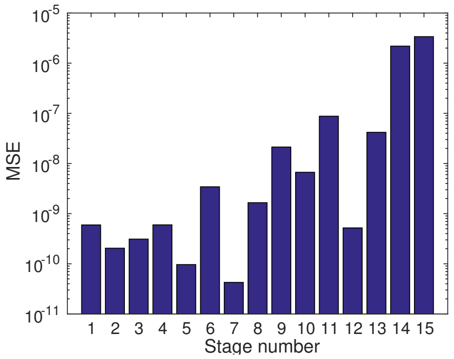

Figure 5: MSE of the first L β encoder($K = 10^5$)

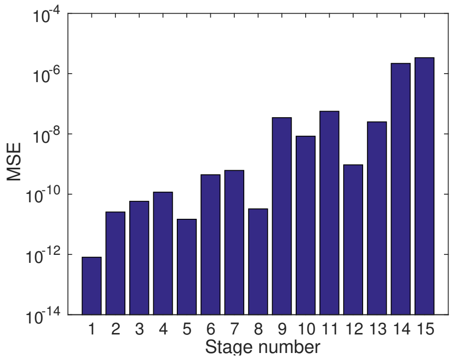

Figure 6: MSE of the first L β encoder($K = 10^7$)

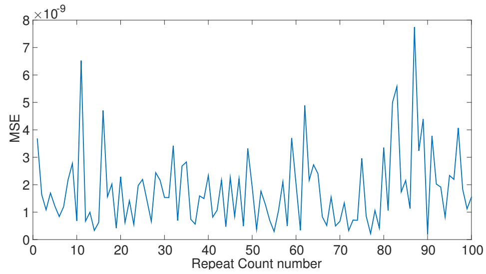

Figure 7: MSE while repeating estimation for 100 times

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