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Control of Avoidance for Chaos by using Downhill Simplex Method

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Abstract—In this study, simple nonlinear dynamical systems are made robust against undesirable bifurcation and unstable states using a downhill simplex method, which is widely used in the optimization problem. The proposed method is based on the concept of robust bifurcation analysis; namely, a stability index defined for each parameter value is updated so as to optimize the index. The optimization method is verified through numerical experiments. The presented results can be generalized for driving a system to avoid chaos.

1. Introduction

In a previous study, robust bifurcation analysis was proposed for automatically determining system parameters [1]. The method used the maximum absolute number of eigenvalues of the linearized dynamics at the considered fixed point as a stability index. By minimizing the stability index, we could determine the optimum system parameters for system robustness. The robust bifurcation analysis is the method that uses the qualitative bifurcation theory based on the dynamical systems theory and the optimization of a performance index in the control theory. However, the stability index is not differentiable with respect to the parameter. In order to solve this problem, the matrix inequality method [2, 3] was proposed, which was based on the control theory. In this direct approach, optimization is achieved with the nonlinear matrix inequality constraint. The objective function and the constraint condition should be smooth with respect to system parameters because the optimization requires the steepest descent direction or the Newton direction of the objective function.

In the present study, we adopt the downhill simplex method [4] to solve the optimization problem with the stability index based on the local expansion rate. This method offers solutions for nonlinear optimization problems, and it does not require differentiability for objective functions because parameters are updated by using objective functions geometrically. Here, we show the results of applying this the method to Hénon map and Kawakami maps, which are known as two-dimensional discrete-time dynamical systems. Furthermore, we demonstrate that we can avoid the chaos observed in these systems and find the parameters for the systems having high stabilities.

2. Stability optimization problem

Now we consider the discrete-time dynamical system described as follows:

$$\begin{aligned} f : R^n \times R^m &\to R^n \\ (x(k), \lambda) &\mapsto x(k+1) = f(x(k), \lambda), \end{aligned} \tag{1}$$

where $\lambda \in R^m$ is a set of system parameters, and $x(k) \in R^n$ denotes internal state variables. k denotes a discrete-time step. The fixed point x^* of map f satisfies $x^* = f(x^*, \lambda)$. The Jacobian matrix $Df(x, \lambda)$ of the map f with respect to the fixed point x^* is defined by

$$Df(x, \lambda) := \frac{\partial}{\partial x} f(x, \lambda)\bigg|_{x=x^*}. \tag{2}$$

As for the stability of non-periodic points x_0 of the map f, we regard x_0 as an N-periodic point, and we use the finite-time Lyapunov exponent, or the local expansion rate, defined by

$$\gamma(N, x_0, \lambda) := \frac{1}{N} \log \|Df^N(x_0, \lambda)\|, \tag{3}$$

for the minimization problem described by

$$\min_{\lambda \in R^m} \gamma(N, x_0, \lambda), \tag{4}$$

where $Df^N(x_0, \lambda)$ is the Jacobian matrix of the N-times map of the map $f(x_0, \lambda)$.

3. Optimization with downhill simplex method

In this study, we use the downhill simplex method to solve optimization problems. The method uses a simple algorithm and does not require a derivative function to optimize the parameters. Therefore, we can use this method for the optimization of non-differentiable objective functions. The algorithm is concretely described as follows: First, consider an $(m + 1)$-polyhedron, whose vertices are corresponding to the parameters in the m-dimensional parametric space. Then, update those vertices iteratively to

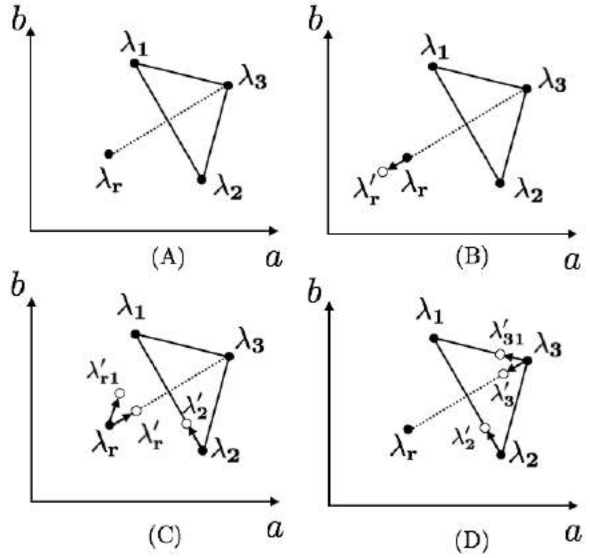


Figure 1: Algorithm of the downhill simplex method.

Algorithm 1 The downhill simplex method with $m = 2$

Require: $\gamma_1 < \gamma_2 < \gamma_3$
Ensure: $\gamma_3 < \theta$
  while $(\gamma_3 > \theta)$ **do**
    if $\gamma_r < \gamma_1$ **then**
      Expand λ_r to the opposite direction from λ_3, and let the point be λ'_{r}. (Fig. 1(B))
      if $\gamma'_{\mathrm{r}} < \gamma_{\mathrm{r}}$ **then**
        $(\lambda_1, \lambda_2, \lambda_3) = \mathrm{swap}(\lambda_1, \lambda_2, \lambda'_r)$
      else
        $(\lambda_1, \lambda_2, \lambda_3) = \mathrm{swap}(\lambda_1, \lambda_2, \lambda_{\mathrm{r}})$
      end if
    else if $\gamma_1 \leq \gamma_{\mathrm{r}} < \gamma_2$ **then**
      $(\lambda_1, \lambda_2, \lambda_3) = \mathrm{swap}(\lambda_1, \lambda_{\mathrm{r}}, \lambda_2)$
    else if $\gamma_2 \leq \gamma_{\mathrm{r}} < \gamma_3$ **then**
      Contract λ_{r} to the direction to λ_3, and let the point be λ'_{r}. (Fig. 1(C))
      if $\gamma'_{\mathrm{r}} < \gamma_r$ **then**
        $(\lambda_1, \lambda_2, \lambda_3) = \mathrm{swap}(\lambda_1, \lambda_2, \lambda'_r)$
      else
        Contract λ_2 and λ_{r} to the direction to λ_1. Let those points be λ'_2 and $\lambda'_{\mathrm{r}1}$. (Fig. 1(C))
        $(\lambda_1, \lambda_2, \lambda_3) = \mathrm{swap}(\lambda_1, \lambda'_2, \lambda'_{\mathrm{r}1})$
      end if
    else if $\gamma_3 \leq \gamma_{\mathrm{r}}$ **then**
      Contract λ_3 to the direction of the midpoint between λ_1 and λ_2, and let the point be λ'_3. (Fig. 1(D))
      if $\gamma'_3 < \gamma_3$ **then**
        $(\lambda_1, \lambda_2, \lambda_3) = \mathrm{swap}(\lambda_1, \lambda_2, \lambda'_3)$
      else
        Contract λ_2 and λ_3 to the direction to λ_1. Let those points be λ'_2 and λ'_{31}. (Fig. 1(C))
        $(\lambda_1, \lambda_2, \lambda_3) = \mathrm{swap}(\lambda_1, \lambda'_2, \lambda'_{31})$
      end if
    end if
  end while

minimize the objective function and converge the objective function to the appropriate value. In this session, the algorithm of the downhill simplex method with $m = 2$ is summarized.

3.1. Algorithm of downhill simplex method

Let a and b stand for the search parameters with $m = 2$. In Fig. 1(A), λ_j, $j = 1, 2$, and 3 denotes the search points, and λ_{r} stands for a reflection point of one of these search points on the (a, b)-parameter plane. γ_j, $j = 1, 2$, and 3, and γ_{r} are the stability indexes, *i.e.*, the local expansion rates at λ_j, $j = 1, 2, 3$ and λ_{r}, respectively. The reflection point λ_{r} is obtained under the point reflection, in the midpoint between λ_1 and λ_2, when $\gamma_1 < \gamma_2 < \gamma_3$ is satisfied. Note that a larger stability index corresponds to a smaller local expansion rate because they have opposite directions. Algorithm 1 shows the algorithm of the downhill simplex method used in this study. θ indicates the threshold for the stability indexes, and the function swap returns the parameters sorted by the descending order in the stability indexes, which is in the ascending order with respect to γ_i.

3.2. Application to discrete-time dynamical systems

The method was applied to 2-dimensional discrete-time dynamical systems, Hénon map and Kawakami map. The dynamics of Hénon map is described by

$$\begin{pmatrix} x(k+1) \\ y(k+1) \end{pmatrix} = \begin{pmatrix} 1 + y(k) - ax(k)^2 \\ bx(k) \end{pmatrix}, \quad (5)$$

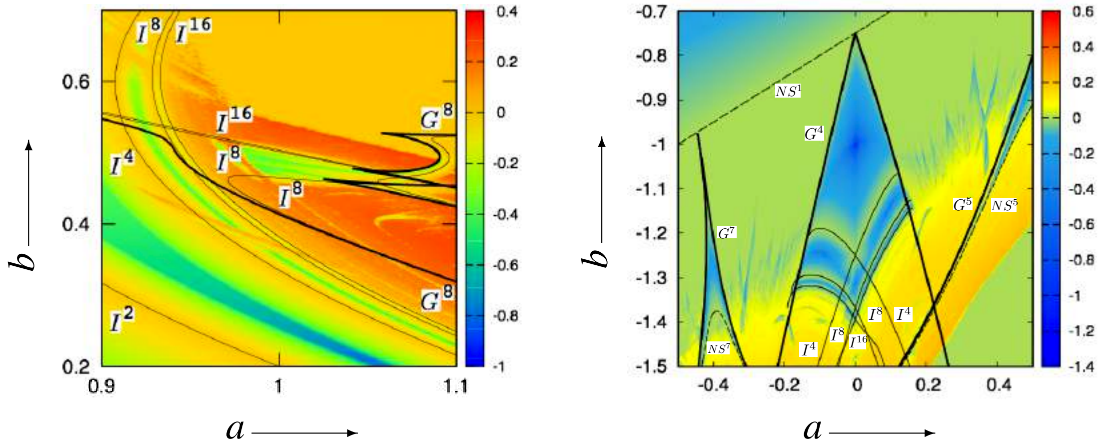
and Kawakami map is described by

$$\begin{pmatrix} x(k+1) \\ y(k+1) \end{pmatrix} = \begin{pmatrix} ax(k) + y(k) \\ x(k)^2 + b \end{pmatrix}, \quad (6)$$

where a and b are system parameters, and x and y indicate internal state variables. First, on the (a, b)-plane, we suppose that a grid spacing of 0.001, let the λ_i^0 at ith grid point be a set of initial parameters. Then, let the λ_{ij}^0, $j = 1, 2$, and 3 at vertices of an equilateral triangle with its gravity at λ_i^0 be a set of initial search points. Periodic points and non-periodic points observed at the search points are assumed to be high order periodic points, and the parameters are updated to the direction minimizing the local expansion rate based on the downhill simplex method.

4. Simulation results

In the bifurcation diagrams shown in this section, G^p, I^p, and NS^p indicate the tangent bifurcation, period-doubling bifurcation, and Neimark-Sacker bifurcation of the periodic points, respectively. Figures 2(a) and (b) show overlapped images of the local expansion rate for attractors and



(a) Hénon map

(b) Kawakami map

Figure 2: Overlapped image of the local expansion rate and the bifurcation diagram for Hénon map and Kawakami map.

the bifurcation diagrams of periodic points of the Hénon map and Kawakami maps. The colored contour plots present the values of the local expansion rate, as indicated by the color bar. Cold color indicates a small local expansion rates and then high stability.

Figure 3 shows a typical simulation result with the bifurcation diagram of the periodic points for the Hénon map. The parameters λ_i^t, obtained with the updates for γ_i^t less than -0.2, are indicated by the small blue dots. The typical trajectories are presented by the red solid lines with arrows. The ends of the line correspond to the initial parameter λ_i^0 and the final parameter λ_i^t. The updates are made in the direction of the arrow. Because the local expansion rate of the bifurcation or chaotic behavior is large and the stability is low, the search points move from those parameter regions to these directions to minimize the local expansion rates. Multiple dots remaining in the shaded region in Fig. 3 exist in the high-stability parameter region, and their local expansion rates are less than -0.2. Note that the search points are not reached by high-stability parameters.

Figures 4(a)–(c) show the simulation results with $\theta = -0.1, -0.3$, and -0.4 for the Hénon map, and Figs. 4(d)–(f) shows those with $\theta = 0, -0.02$, and -0.1 for the Kawakami map. The red colored small dots indicate the arrival points at parameter λ_i^t. The fact that the small blue or red dots in Figs. 3 and 4 are mainly distributed in the region with the negative local expansion rate in Fig. 2 suggests that our method successfully operates the system to avoid chaos. In addition, choosing the threshold θ, we could control the stabilities of the systems to avoid chaos and low-stability conditions.

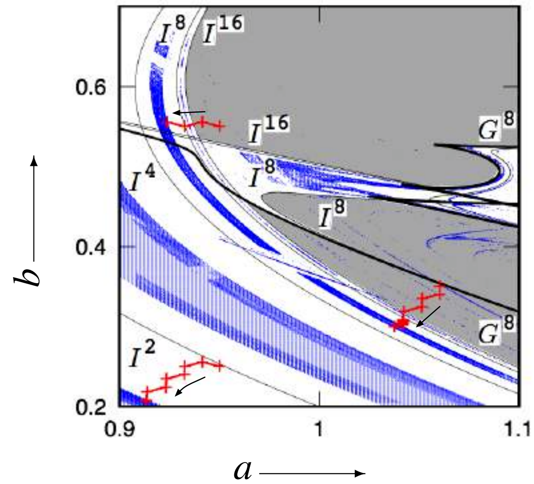


Figure 3: Chaos avoidance in the case of Hénon map. $(\gamma_i^t \leq -0.2)$

5. Conclusion

In this study, robustification of a nonlinear dynamical system is considered, and the downhill simplex method is applied to solve the optimization problems in robust bifurcation analysis for dynamical systems based on the Hénon map and Kawakami maps. The advantage of this method is that it does not require differentiability of the objective functions. The method is shown to be efficient, and it can be generalized for avoiding chaos.

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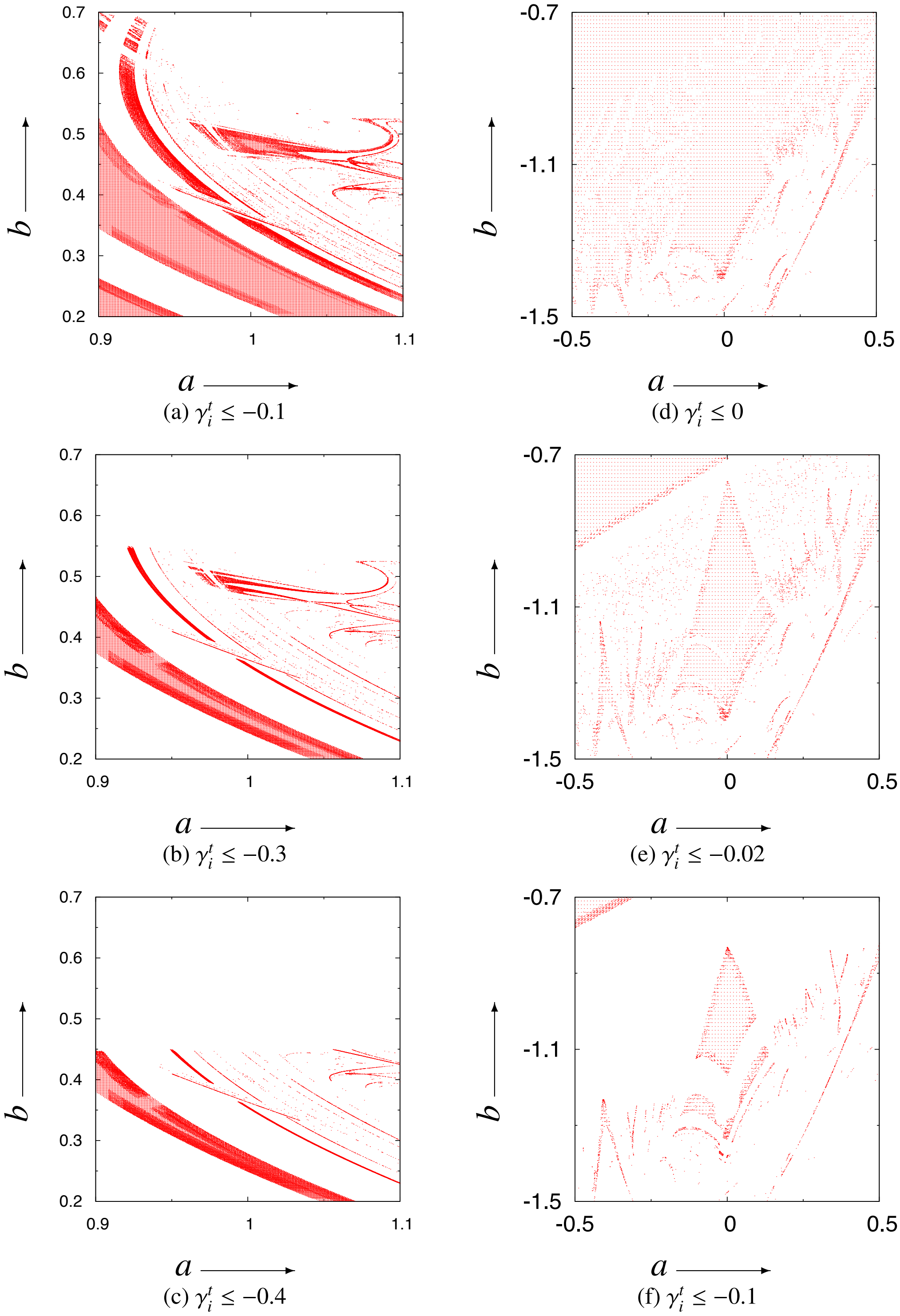

Figure 4: Chaos avoidance with downhill simplex method in the case of (a)–(c):Hénon map and (d)–(f): Kawakami Map.

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