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A spiking neuron model with two slopes and triangular wave base signal

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Abstract—This paper considers a spiking neuron model with two slopes and triangular wave base signal. The state of this neuron repeats a integrate-and-fire dynamics and the slope of the state is alternatively changed at every resetting. We focus on the timing of the spike phase and derive the spike phase map. Using the map, we consider behavior of the neuron. It is shown that the spiking neuron exhibits not only chaos and periodic phenomena but also the superstable and co-existence phenomena.

1. Introduction

This paper considers a spiking neuron model with two slopes and triangular wave base signal. The state variable of this spiking neuron operates a integrate-and-fire dynamics. The state has a constant slope and rises with time. If the state reaches the threshold line, the neuron outputs a spike and the state resets to the triangular wave base signal where the resetting and output are instantaneous and simultaneous. Next, the state rises with another constant slope. After reaching the threshold and resetting to the base signal, the state again rises with the first slope. Repeating above dynamics, the neuron exhibits nonperiodic and periodic trajectory and corresponding to the output of the spike.

In order to consider the generating phenomenon, we focus on the timing of the spike and derive the spike phase map. Since this map is the return map and piecewise linear, it has advantage that we can analyze rigorously. When the parameters vary, the shape of the map, the slope of the segment and the number of discontinuous points change. Using the map and Lyapunov exponent, we show that the neuron generates various phenomena. In addition, the map exhibits the superstable periodic orbit (SSPO) [1] with various period and co-existence phenomena of chaos and SSPO. These phenomena are also shown, using the map.

Many papers have been studied artificial neuron models [2]-[4] and the spiking neuron is a kind of the model. The spiking neuron exhibits various nonlinear phenomena as shown in this paper though a simple model. Ref. [5] has studied for bifurcation phenomena. Using the spike-output and two neurons, the pulse-coupled system has been studied in Ref. [6]. These investigation results are basic for consideration of synchronization phenomenon. In addition, applications for the analog-to-digital converters have been considered to encode the timing of the spike [7]-[8]. We think that discussion and consideration in this paper are approach to advance in respect of above investigation.

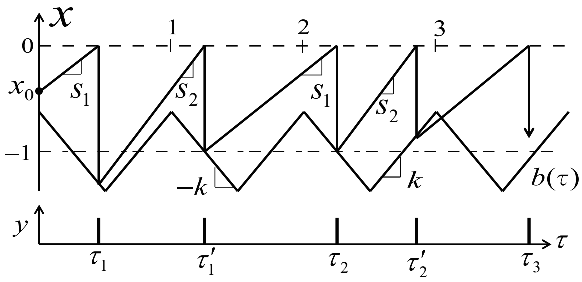

Figure 1: Basic dynamics of the spiking neuron.

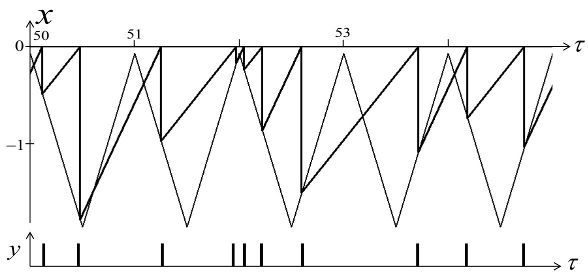

Figure 2: An example of the waveform and output. $s_1 = 2.4, s_2 = 1.7, k = 3.7$.

Previous work [5] has studied the spiking neuron model with triangular wave base signal. However the slope of the state in Ref. [5] is fixed one constant slope.

2. Basic dynamics and typical phenomena of the neuron

The dynamics of targeted spiking neuron in this paper is described by Equation (1).

$$\begin{cases} \frac{dx}{d\tau} = s_1,\ y = 0 & \text{for } x < 0 \text{ and } l \text{ is even,} \\ \frac{dx}{d\tau} = s_2,\ y = 0 & \text{for } x < 0 \text{ and } l \text{ is odd,} \end{cases} \quad (1)$$
$$x(\tau+) = b(\tau+),\ y(\tau+) = 1, \ \text{ if } \ x \geq 0,$$

where τ, x and y are dimensionless time, state variable and output of the neuron. Let l be the number of times at which the state hits threshold level $x = 0$ and l is nonnegative integer. The base signal $b(\tau)$ is triangular waveform with period 1 as follows.

$$b(\tau + 1) = b(\tau),$$
$$b(\tau) = \begin{cases} -k(\tau - \frac{1}{4}) - 1 & \text{for } \ 0 \leq \tau < 0.5, \\ k(\tau - \frac{3}{4}) - 1 & \text{for } \ 0.5 \leq \tau < 1. \end{cases} \quad (2)$$

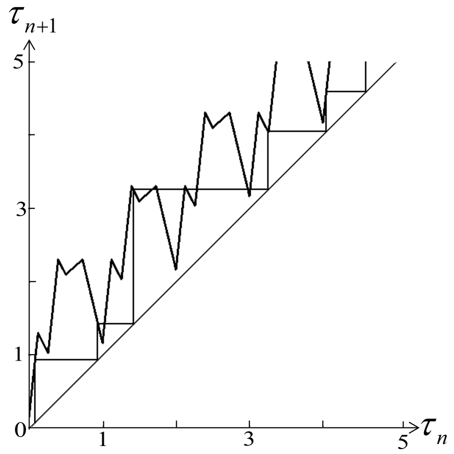

Figure 3: Spike position map. $s_1 = 2.4, s_2 = 1.7, k = 3.7$.

This neuron has three parameters: slopes of the state variable s_1, s_2 and slope of the base signal k. s_1, s_2, k are real parameters and we restrict to $0 < s_1, 0 < s_2, 0 < k < 4$, and the initial state $-1 < x(0) < 0$. An example of the dynamics is shown in Fig. 1. At first, the state $x(\tau)$ starting from the initial state $x(0) = x_0$ rises with time where the slope is s_1. If x reaches the threshold line $x = 0$, the spike y is outputted and the state x resets to the base signal $b(\tau)$. For simplicity, we assume that the spike-output and resetting are instantaneous and simultaneous. Next, x rises with another slope s_2. If x again reaches the threshold line, the spike is outputted and x resets to the base $b(\tau)$ instantaneously. The slope of the state is alternately changed to s_1 and s_2 at every resetting. The spiking neuron repeats above dynamics. Fig. 2 shows an example of the time waveform. We note that Ref. [5] have discussed the spiking neuron with fixed one constant slope ($s = s_1 = s_2$) and studied for bifurcation phenomena.

In order to consider the behavior of this neuron, we define the spike position map. As shown in Fig. 1, let τ_n and τ_{n+1} (or τ'_n and τ'_{n+1}) be n-th and $n+1$-st spike positions of odd numbers (or even numbers). The spike position τ_n determines next spike position τ'_n and τ'_n determines the spike position τ_{n+1}. Since τ_{n+1} depends on τ_n, the spike position map F can be defined as follows.

$$\begin{aligned} \tau_{n+1} &\equiv F(\tau_n) = F_1(F_2(\tau_n)), \\ \tau_{n+1} &\equiv F_1(\tau'_n) = \tau'_n - \tfrac{1}{s_1} b(\tau'_n), \\ \tau'_n &\equiv F_2(\tau_n) = \tau_n - \tfrac{1}{s_2} b(\tau_n), \end{aligned} \tag{3}$$

where functions F_1 and F_2 are described theoretically and are the same as those of the system in Ref. [5]. An example of the spike position map is show in Fig. 3. This map is piecewise linear and the shape is varied by changing the parameters.

Here we introduce the phase θ as new variable and define the spike phase $\theta_n \equiv \tau_n$ mod1 and $\theta'_n \equiv \tau'_n$ mod1. That is, θ_n and θ'_n denote decimals of τ_n and τ'_n. We can consider the

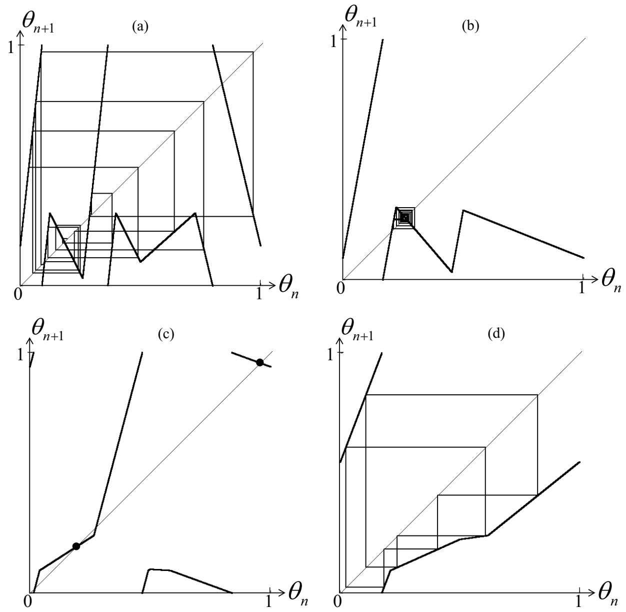

Figure 4: Typical spike phase map where $s_1 = 2.4$. (a) Chaotic orbit for $s_2 = 1.4$ and $k = 3.7$. (b) Chaotic orbit for $s_2 = 3.2$ and $k = 3.7$. (c) Stable fixed points for $s_2 = 1.4$ and $k = 1.7$. (d) Periodic orbit for $s_2 = 3.2, k = 1.7$.

same as the spike position map for the spike phase because θ_{n+1} depends on θ_n. Therefore, we define the spike phase map f as follows.

$$\begin{aligned} \theta_{n+1} &= f(\theta_n) = g_1(\theta'_n) \bmod 1, \\ \theta'_n &= g_2(\theta_n) \bmod 1. \end{aligned} \tag{4}$$

$$\begin{aligned} g_1(\theta'_n) &= \begin{cases} (1 + \frac{k}{s_1})\theta'_n + \frac{1}{s_1}(1 - \frac{k}{4}) & \text{for } 0 \le \theta'_n < 0.5 \\ (1 - \frac{k}{s_1})\theta'_n + \frac{1}{s_1}(1 + \frac{3k}{4}) & \text{for } 0.5 \le \theta'_n < 1; \end{cases} \\ g_2(\theta_n) &= \begin{cases} (1 + \frac{k}{s_2})\theta_n + \frac{1}{s_2}(1 - \frac{k}{4}) & \text{for } 0 \le \theta_n < 0.5 \\ (1 - \frac{k}{s_2})\theta_n + \frac{1}{s_2}(1 + \frac{3k}{4}) & \text{for } 0.5 \le \theta_n < 1. \end{cases} \end{aligned} \tag{5}$$

Functions g_1 and g_2 are described theoretically and are the same as those of the system in Ref. [5]. Fig. 4 shows examples of the spike phase map. Although the spike phase map is piecewise linear, discontinuous points appear because of taking modulus for the spike phase. The parameters of Fig. 4 (a) correspond to those of Fig. 3.

In order to consider behavior of the spike phase map, we describe some definitions as follows.

Definition1: A point θ_n is said to be a fixed point if $\theta_n = f(\theta_n)$. A point θ_n is said to be a periodic point with period k if $\theta_n = f^k(\theta_n)$, $\theta_n \neq f^j(\theta_n)$ for $1 \le j < k$ where f^k denotes the k-fold composition of f and $k > 1$. The periodic points with period k are said to be unstable (or stable) for initial state if $|Df^k(\theta_n)| > 1$ (or $|Df^k(\theta_n)| < 1$), where $Df \equiv \frac{d}{d\theta_n} f$. An orbit of the stable periodic points is referred to as stable periodic orbit. If $f(I) \subseteq I$ and there exists some positive integer l such that $|Df^l(\theta_1)| \ge 1$ for almost all $\theta_1 \in I$ where $I \equiv [0, 1]$, then the orbit is unstable and the map f is said to generate chaos [9].

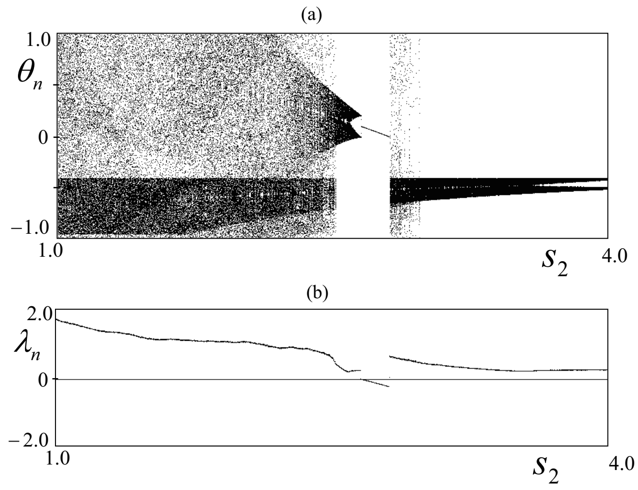

Figure 5: Characteristics for changing of s_2 where s_1 = 2.4 and k = 3.7. (a) Bifurcation diagram. (b) Lyapunov exponent.

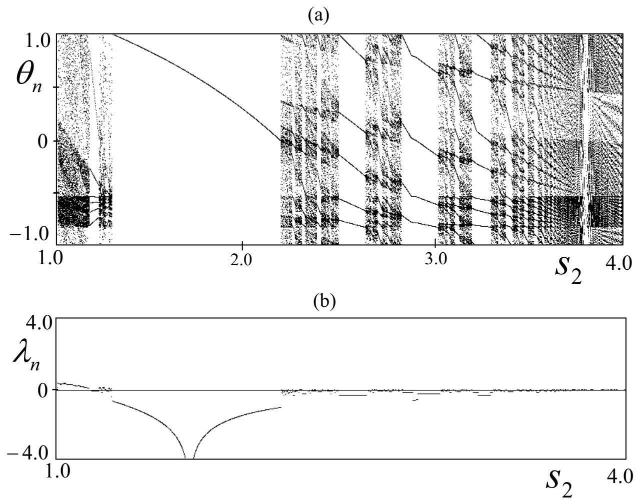

Figure 6: Characteristics for changing of s_2 where s_1 = 2.4 and k = 1.7. (a) Bifurcation diagram. (b) Lyapunov exponent.

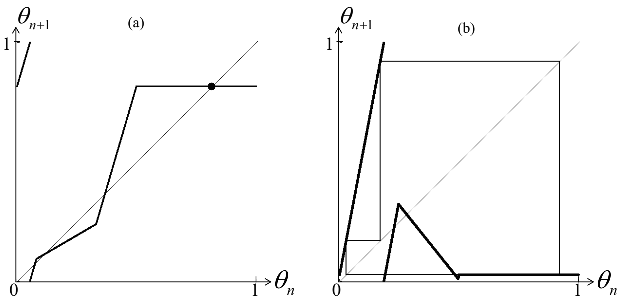

Figure 7: Examples of the SSPOs where s_1 = 2.4. (a) s_2 = 1.7 and k = 1.7. (b) s_2 = 3.9 and k = 3.9.

Fig. 4 (a) and (b) exhibit chaotic orbits. Fig. 4 (c) has a fixed point (black circle) and exhibits periodic orbit with period 1. Fig. 4 (d) exhibits periodic orbit with period 7. The map exhibits a variety of behavior by changing the parameters.

We show bifurcation diagram and Lyapunov exponent λ_n for s_2 in Fig. 5 when $s_1 = 2.4$ and $k = 3.7$ are fixed, where this figure corresponds to Fig. 4 (a) and (b). In these parameters, The range which has positive Lyapunov exponent is wide: the range generating chaotic orbit is wide. As shown in Fig. 4 (a) and (b), when the parameter k is large, the slope of the segment on the map becomes steep and the orbit tends to be unstable. Therefore, the map tends to exhibit chaos.

Possible range which value of θ_n takes is wide in the neighborhood of $s_2 = 1.5$, however possible range is narrow in the neighborhood of $s_2 = 3.5$. θ_n is timing of the spike-output. In the neighborhood of $s_2 = 1.5$, it is shown that spike timings spread out in the range $[0, 1]$. In the neighborhood of $s_2 = 3.5$, it is shown that spike timings concentrate in a local region. Although both these phenomena are qualitatively chaotic, we can see that characteristics of the spike-output are different. In the spiking neuron model, the analog input is encoded through the spike timing and applications to the analog-to-digital converter have been studied in Refs. [7] and [8]. We think that investigation of the spike phase and spike-output characteristics are important but it is in future works.

Fig. 6 shows bifurcation diagram and Lyapunov exponent λ_n for s_2 when $s_1 = 2.4$ and $k = 1.7$ are fixed, where this figure corresponds to Fig. 4 (c) and (d). When $k = 1.7$, the map has some parts of gradual slope as shown in Fig. 4 (c) and (d) because k is small. Therefore, the orbit tends to become stable and tends to exhibit periodic orbit. In addition, various periodic orbits exist by changing the parameters and the range which has negative Lyapunov exponent is wider as compared with Fig. 5.

3. Various interesting phenomena

In Section 2, we show that this spiking neuron exhibits a variety of chaotic and periodic phenomena. In this section, more interesting phenomena are shown in using the spike phase map.

3.1. Superstable periodic orbits (SSPOs)

If the parameters $s_1 = k$ or $s_2 = k$ is fixed, the spike phase map has a flat part: there is the segment with the slope 0. Typical spike phase map is shown in Fig. 7. Here we describe the definition in respect of above as follows.

Definition2: The periodic points with period k are said to be superstable for initial state if $|Df^k(\theta_n)| = 0$. An orbit of the superstable periodic points is referred to as superstable periodic orbit (SSPO).

In Fig. 7 (a), the map has a fixed point with the slope 0 and therefore exhibits the superstable fixed point (or SSPO

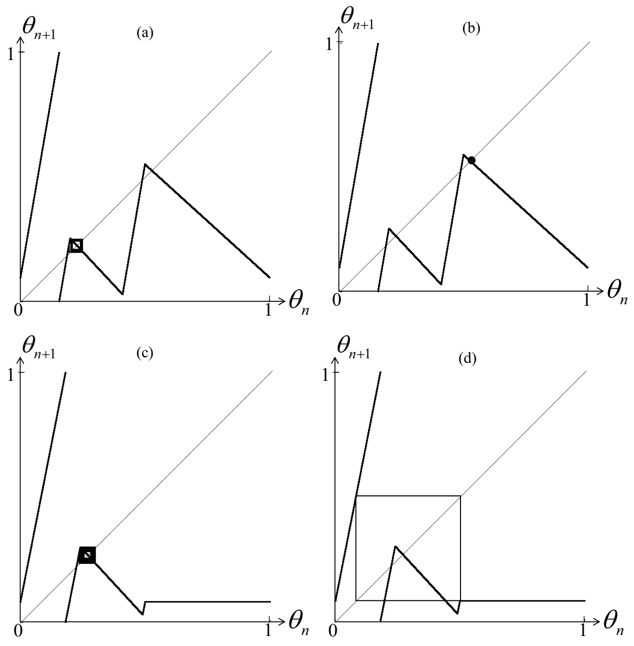

Figure 8: Examples of the co-existence phenomena where $k = 3.7$. (a) and (b) are the co-existence phenomena of chaos and stable fixed point. (c) and (d) are the co-existence phenomena of chaos and SSPO. (a) $s_1 = 2.55, s_2 = 2.7$ and $x(0) = -0.8$. (b) $s_1 = 2.55, s_2 = 2.7$ and $x(0) = -0.3$. (c) $s_1 = 2.4, s_2 = 3.7$ and $x(0) = -0.1$. (d) $s_1 = 2.4, s_2 = 3.7$ and $x(0) = -0.8$.

with period 1). We note that Lyapunov exponent is negative infinity when the map exhibits the SSPO. In the steady state for Fig. 7 (b), we can see that the orbit starting from the flat segment returns to the flat segment. This map exhibits the SSPO with period 3. This neuron can generate various SSPOs by changing the parameters.

3.2. Various co-existence phenomena

Although the parameters of the system are same, for different initial state, the system exhibits different behavior in the steady state. This phenomenon is called co-existence phenomenon. Examples of the co-existence phenomenon are shown in Fig. 8. Fig. 8 (a) and (b) are the same parameters $s_1 = 2.55, s_2 = 2.7$ and $k = 3.7$, however the initial states are different. The initial state is $x(0) = -0.8$ in Fig. 8 (a) and is $x(0) = -0.3$ in Fig. 8 (b). We can see that Fig. 8 (a) exhibits chaotic orbit and Fig. 8 (b) exhibits periodic orbit (stable fixed point) in the steady state. Therefore, this neuron exhibits co-existence phenomenon of chaotic and periodic orbit for $s_1 = 2.55, s_2 = 2.7$ and $k = 3.7$. Fig. 8 (c) and (d) also have the same parameters $s_1 = 2.4, s_2 = 3.7$ and $k = 3.7$ and the initial states are different. We can see that Fig. 8 (c) exhibits chaotic orbit, however the orbit in Fig. 8 (b) hits the flat segment and the SSPO with period 2 is generated. In this case, this neuron exhibits co-existence phenomenon of chaos and SSPO for $s_1 = 2.4, s_2 = 3.7$ and $k = 3.7$. In Fig. 7 (a), we can see that the map has the superstable fixed point (black circle) while also has stable fixed point in the lower left part. In this case, this neuron exhibits co-existence phenomenon of the SSPO and periodic orbit. Proposed our spiking neuron can exhibit various co-existence phenomena and we think that very complicated bifurcation phenomena occur.

4. Conclusions

In this paper, a spiking neuron model with two slopes and triangular wave base signal has been studied. We derive the spike phase map and consider generating phenomena. By changing the parameters the spiking neuron exhibits chaos and periodic phenomena. We can see that the neuron exhibits various SSPOs and co-existence phenomena for some parameters. In the future, we intend to analyze bifurcation phenomena and consider characteristics of the spike-output.

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