Accuracy Improvement in FEM Solution of Waveguide Discontinuity Problems by Dual E and H Formulations*

Ting-Huei Lin[1] and Ruey-Beei Wu[2], senior Member, IEEE

1. Telecommunication Laboratories, Chunghwa Telecommunications Co., Ltd.
2. Department of Electrical Engineering and Graduate Institute of Telecommunications Engineering, National Taiwan University, Taipei, Taiwan 106, R.O.C.
E-mail: rbwu@ew.ee.ntu.edu.tw

I. INTRODUCTION

The finite element method (FEM) has become a very important numerical technique in solving waveguide discontinuity problems, with great capability and versatility in handling complex and arbitrary geometries. Although the rapid advances in computers impose less severe restrictions on these problems, the requirement for more accurate solution calls for finer meshes or higher order of the shape function, either one implying a dramatic increase in computation time. Some researchers have focused on adaptive mesh refinement to improve the accuracy of FEM by refining local critical regions instead of global regions [1], [2].

The paper exploits another scheme to improve the solution accuracy of waveguide discontinuity problems. The FEM analysis can be formulated in terms of electric or magnetic fields, called E or H formulations [3], [4]. Connecting waveguides of arbitrary cross section can be truncated by modal expansion [5] or PML [6]. Solving the same problem with the same mesh, the two formulations yield two different results. As it had been pointed out [7], [8] that by analyzing for both E and H, useful information on the errors in the solution can be obtained. It is interesting to take the ideas a step further to combine the two results obtained by the dual formulations for a better solution.

II. DUAL FORMULATION

For the electromagnetic scattering off volumetric, metallic waveguide discontinuities, the electric field in the junction region can be shown to satisfy the variational equation [5]

$$\delta L_1 = 0;$$

$$L_1(\vec{E}) = -\frac{1}{2}\int_{\Omega}(j\omega\varepsilon\vec{E}\cdot\vec{E} + \frac{1}{j\omega\mu}\nabla\times\vec{E}\cdot\nabla\times\vec{E})d\Omega + \frac{1}{2}\sum_i\sum_{n=1}^{\infty}\int_{\Gamma_i}\vec{E}\times\vec{h}_{tn}\cdot\hat{n}d\Gamma \int_{\Gamma_i}\vec{E}\times\vec{h}_{tn}\cdot\hat{n}d\Gamma$$
$$-2\sum_i\sum_{n=1}^{\infty}a_n^+\int_{\Gamma_i}\vec{E}\times\vec{h}_{tn}\cdot\hat{n}d\Gamma \quad (1)$$

The equation is called E-formulation, for which the electric fields are chosen as unknowns. Similarly, one can derive the H-formulation by choosing magnetic fields as unknowns, i.e.,

$$\delta L_2 = 0;$$

$$L_2(\vec{H}) = -\frac{1}{2}\int_{\Omega}(j\omega\mu\vec{H}\cdot\vec{H} + \frac{1}{j\omega\varepsilon}\nabla\times\vec{H}\cdot\nabla\times\vec{H})d\Omega + \frac{1}{2}\sum_i\sum_{n=1}^{\infty}\int_{\Gamma_i}\vec{e}_{tn}\times\vec{H}\cdot\hat{n}d\Gamma \int_{\Gamma_i}\vec{e}_{tn}\times\vec{H}\cdot\hat{n}d\Gamma$$
$$-2\sum_i\sum_{n=1}^{\infty}a_n^+\int_{\Gamma_i}\vec{e}_{tn}\times\vec{H}\cdot\hat{n}d\Gamma \quad (2)$$

In the special case that there is only one fundamental mode of unit amplitude incident on port 1, the functional L_1 in (1) is a stationary formula for $-S_{11}-1$, while L_2 in (2) a stationary formula for $S_{11}-1$ [5].

It is a typical procedure to discretize (1) and (2) for numerical computation by FEM. Here we adopt the edge-based method and tetrahedral elements for 3-D problems. The solution region is divided into

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small tetrahedra and the fields are expanded by the Whitney one-form edge basis with the tangential field components along the edges as the expansion coefficients. As far as the scattering parameters are concerned, only these unknowns along the boundary are relevant. The matrix equation is solved by the frontal solution technique [5]. The final boundary matrix, being independent of the excitation on the waveguide ports, is LU-decomposed.

From the solved tangential fields along the boundary, the unknown reflection coefficients a_n^- can also be obtained from E-formulation as

$$a_n^- = \int_{\Gamma_i} \vec{E} \times \vec{h}_{tn} \cdot \hat{n} d\Gamma - a_n^+ \tag{3}$$

or from H-formulation as

$$a_n^- = a_n^+ - \int_{\Gamma_i} \vec{e}_{tn} \times \vec{H} \cdot \hat{n} d\Gamma \tag{4}$$

III. NUMERICAL RESULTS

A. One-dimensional homogeneous waveguide

In order to find the relationship between the S_{11} derived from E- and H-formulations under the same meshes, we consider the simplest infinite parallel plate waveguide shown in the inset of Fig.1. If the material in this waveguide is homogeneous, there will obviously be no reflected waves between port 1 and port 2. But when we solve the problem by FEM method, we will get some reflected waves induced from discretization error, which is mainly caused from approximating the sinusoidal field distribution in the waveguide by piecewise linear functions in the FEM solution process.

In this special simple 1-D case, the numerical solution of E-field got from E-formulation (1) and H-field got from H-formulation (2) can be related directly as $\vec{E} = \eta \vec{H} \times \hat{z}$. It is not difficult to verify that the reflection coefficient from (3) and (4) are related by

$$a_n^-\Big|_{E_{form}} = \int_{\Gamma_i} \vec{E} \times \vec{h}_{tn} \cdot \hat{n} d\Gamma - a_n^+ = -(a_n^+ - \int_{\Gamma_i} \vec{e}_{tn} \times \vec{H} \cdot \hat{n} d\Gamma) = -a_n^-\Big|_{H_{form}} \tag{5}$$

where the subscript of a_n^- denotes the result obtained from E- or H-formulation. This means that the discretization error incurred a_n^- from E- and H-formulations are of the same magnitude and opposite signs. If the results are added and averaged directly, the discretization error will be cancelled exactly! The numerical results of reflection coefficient with division size as a parameter are shown in Fig. 1 to demonstrate the interesting relationship.

B. Rectangular waveguide filled with step-index material

For general three-dimensional cases, E- and H-formulations call for different modeling for PEC boundary and discontinuity regions. The relationship (5) fails to exist and the corresponding expression for the discretization error a_n^- got from E- and H- formulations cannot be found now.

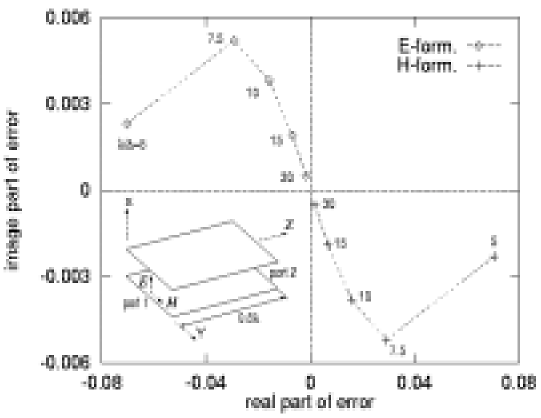

Fig. 1. Numerical error of calculated S_{11} for one-dimensional waveguide with mesh division as a parameter.

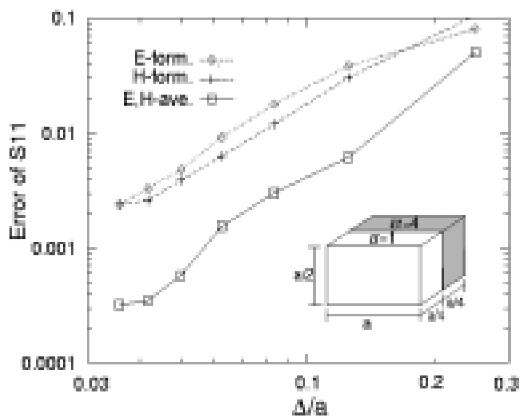

Fig. 2. Numerical error of calculated S_{11} for rectangular waveguide filled with a step-index dielectric at a/λ=0.6.

To test the accuracy improvement for 3-D FEM by averaging the results from both formulations, we demonstrate it by direct numerical simulation. Consider a simple rectangular waveguide filled with step-index material, for which exact solution is available. Fig.2 shows the absolute error in S_{11} got from FEM with dual formulations and from their averaging with the same meshes. It is found that the accuracy got from the two formulations is about the same but opposite signs. After taking the average of the results from these two formulations, the accuracy is improved by about one order of magnitude and the convergence remains quadratic.

Similar analysis has been employed to calculate S_{21} at different ports as shown in Fig. 3. The calculated results by the dual formulations can still give a good estimate of the solution accuracy. However, the simple averaging of the dual formulation results cannot yield as successful error reduction as it can to S_{11}.

The same approach is applied for the same problem but with port 2 terminated with a perfectly matched layer followed by PEC. The permitivity and permeability of PML are [6]: $\varepsilon = \varepsilon_0 \varepsilon_r [\Lambda]; \mu = \mu_0 \mu_r [\Lambda]$ where $[\Lambda] = \mathrm{Diag}\{1\text{-}3j,\ 1\text{-}3j,\ 1\text{+}.3j\}$. The PML is lossy, anisotropic, and its length is chosen to be *a/4* in the simulation. Theoretically it will attenuate the input power by more than 80 dB. Fig. 4 shows that even in the inhomogeneous, lossy, and anisotropic case, averaging the results got from the E- and H-formulations can also improve the accuracy significantly.

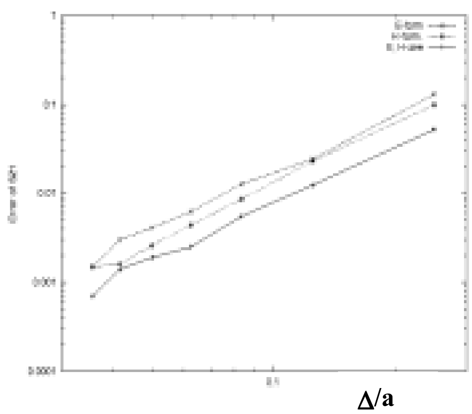

Fig. 3. Numerical error of calculated S_{21} (TE_{01} mode) for rectangular waveguide filled with a step-index dielectric at *a*/λ=0.6.

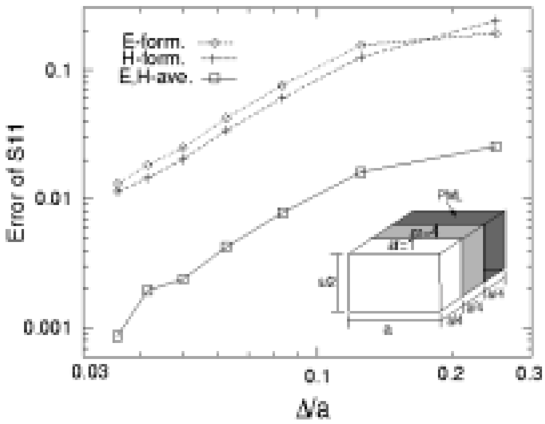

Fig. 4. Numerical error of calculated S_{11} for rectangular waveguide filled with a step-index dielectric at *a/*λ=0.6 and terminated by conductor backed PML.

C. Dielectric obstacle in rectangular waveguide

Fig. 5 shows another example of a rectangular waveguide loaded with a dielectric obstacle [4, Fig. 3.]. In this case, not only propagating waves but also evanescent fields will be excited near the discontinuity region. The geometric parameters are chosen to be the same as those in [4]. In the present analysis, the FEM results based on E- and H-formulations with different division sizes and the average of both formulations are compared. It is clearly demonstrated that the reflection coefficients obtained by the average of both formulations with a coarser mesh exhibit better accuracy than those by using either E- or H-formulation only.

D. Rectangular to dielectric-filled circular waveguide transition

We may also verify the present method for a practical waveguide transition design between a rectangular and dielectric-filled circular waveguides [5, Fig. 7]. FEM method based on E- and H-formulations with different division sizes and the average of both formulations are used to compare convergence of the results. Fig. 6 demonstrates again that the reflection coefficients obtained by the average of both formulations with coarser mesh exhibit better convergence than those by using either E- or H-formulation only.

IV. CONCLUSIONS

The method hybridizing FEM and modal expansion technique are employed to solve metallic waveguide discontinuity problems. The solution can be based on either E- or H-formulation with the same mesh division and an almost identical numerical procedure. It is found that the discretization errors of the results calculated by the two dual formulations are roughly of the same order. By taking the average of the calculated S_{11} from both E- and H-formulations with the same meshes, the accuracy can be dramatically improved.

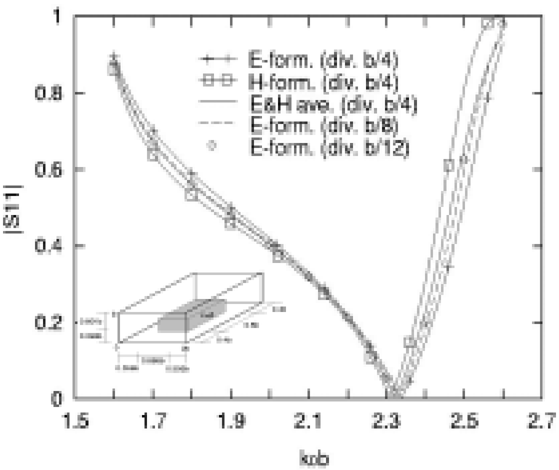

Fig. 5. Comparison among FEM results based on E-, H-formulations and their average with mesh division as a parameter for a dielectric loaded rectangular waveguide.

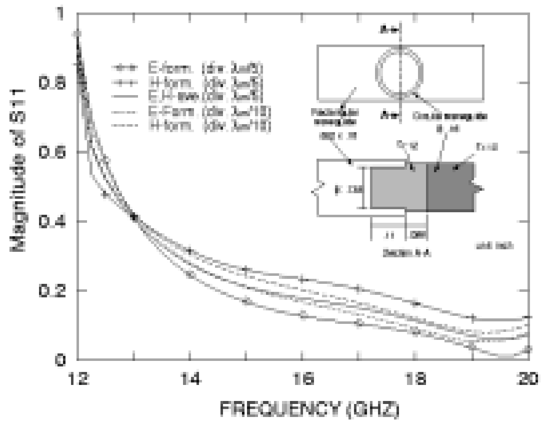

Fig. 6. Comparison among FEM results for calculated reflection coefficients of rectangular to dielectric-filled circular waveguide transition.

In our test cases, the average results can have the accuracy about one order of magnitude higher, with roughly the same convergence rate versus discretization cell size. It is also demonstrated that the method can not only be used in isotropic inhomogeneous regions, but also in anisotropic PML material. This implies that the method can be advantageous not only for metallic waveguide discontinuity problems but also for electromagnetic radiation problems in an unbounded region.

However, for the test cases tried, averaging of the dual formulation for scattering parameters other than S_{11} did not lead to any significant error reduction.

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